Two results on divisors on moduli spaces of sheaves on algebraic surfaces: generic Strange
Duality on abelian surfaces and Nef cones of Hilbert schemes of points on surfaces with
irregularity zero

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"Mihi vel tellus optem prius ima dehiscat

vel pater omnipotens adigat me fulmine ad umbras,

pallentis umbras Erebo noctemque profundam,

ante, pudor, quam te violo aut tua iura resolvo.

Ille meos, primus qui me sibi iunxit, amores

abstulit; ille habeat secum servetque sepulcro"

Virgil, Aeneids book IV

To Music, the love of my life

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Abstract of Dissertation

In the first part of this thesis, we consider a special version of Le Potier's strange duality conjecture for sheaves over abelian surfaces, after other two versions were studied in previous literature. In the current setup, the isomorphism involves moduli spaces of sheaves with fixed determinant and fixed determinant of the Fourier-Mukai transform on one side, and moduli spaces where both determinants vary, on the other side. We first establish the isomorphism in rank one using the representation theory of Heisenberg groups. For product abelian surfaces, the isomorphism is then shown to hold for sheaves with fiber degree 1 via Fourier-Mukai techniques. By degeneration to product geometries, the duality is obtained generically for a large number of numerical types. Finally, it is shown in great generality that the Verlinde sheaves encoding the variation of the spaces of theta functions are locally free over moduli. In the second part, we discuss general methods for studying the cone of ample divisors on the Hilbert scheme of n points $X^{[n]}$, where X is a smooth projective surface of irregularity 0. We then use these techniques to compute the cone of ample divisors on $X^{[n]}$ for several surfaces where the cone was previously unknown. Our examples include families of surfaces of general type and del Pezzo surfaces of degree 1. The methods rely on Bridgeland stability and the Positivity Lemma of Bayer and Macrì.

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Disclaimer

I hereby declare that the work in this thesis is that of the candidate alone, except where indicated in the text, and as described below.

Chapters 2 to 6 are a joint work with Dr. Alina Marian, Dr. Dragos Oprea and Dr. Kota Yoshioka, to be published in the Journal of Algebraic Geometry as *On the strange duality conjecture for abelian surfaces, II*, 37 pages, available at `arXiv:1402.6715`.

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Introduction

This thesis is articulated in two parts, both of which are related to the study of divisors on moduli spaces of sheaves over surfaces. The first parts deals with the study of certain divisors, known as Theta divisors, which arise naturally via a determinantal construction. Their most prominent feature is that, in some cases of interest, they exhaust all the possible isomorphism classes of divisors on the moduli space, thus covering its entire Picard group. The relationship between the spaces of sections of the Theta divisors will be our focus for the first part of this thesis: it was observed, first in the case of curves and then in the case of surfaces by Le Potier, that the spaces of section of some of these divisors are subject to certain natural dualities. It was then conjectured that such duality phenomenon should hold in a vast generality: the case of our interest will be that of moduli spaces over abelian surfaces, and our main result will show how in this case one can show that such conjecture holds for a generic abelian surface.

The second part of this thesis explores the possibility to describe the Nef cones of Hilbert schemes of points on surfaces using a powerful tool which was constructed by Bridgeland in 2006: a notion of stability for complexes of sheaves which generalizes the classical slope stability for sheaves on algebraic curves, but proves different in most other cases. The study of Bridgeland stability conditions has been shown to have applications in many fields of mathematics, and the phenomenon we are interested in is studying is a deep relationship between Bridgeland stability conditions on algebraic surfaces an Nef divisors, which was first established by Bayer and Macrí in 2014. We will exploit such relationship and apply it in a

new case of interest, and by doing so we will be able to give a complete description of the Nef cone of the Hilbert schemes of points on a large class of surfaces.

We will now give a more detailed introduction to each of the two topics to organize the structure of this thesis.

0.1 Strange Duality and Verlinde numbers

Let (X, H) be a complex projective polarized curve or surface, $v \in K(X)$ be a class in the topological K-theory $K(X)$, and let us denote by $\mathfrak{M}_v$ the moduli space of Gieseker H-stable sheaves on X with Mukai vector v. Consider a Mukai vector w, orthogonal to v with respect to the Euler form $(v, w) := \chi(v \otimes w)$ on $K(X)$. There is a group homomorphism

$$\Theta : v^\perp \longrightarrow \operatorname{Pic}(\mathfrak{M}_v) \ , \ w \mapsto \Theta_w \tag{1}$$

considered in [Dre87, LP92, Li96]. The *Theta line bundle* $\Theta_w \to \mathfrak{M}_v$ is obtained by a standard determinantal construction, see [LP92, Li96]. In some cases of importance, the group homomorphism (1) is an isomorphism, hence the Picard group of the moduli space can be completely described in terms of Theta divisors. Of course, since the orthogonality condition on v and w is symmetric, we can construct a theta line bundle Θ_v on $\mathfrak{M}_w$ in the same fashion. In fact, if one takes any two classes v and w satisfying $(v, w) = 0$, under easily achievable assumptions, the jumping locus

$$\Theta = \{(E, F) \in \mathfrak{M}_v \times \mathfrak{M}_w \mid H^0(E \otimes F) \neq 0\}$$

should be divisorial, hence its sheaf of ideals $\mathcal{O}(\Theta)$ should be a line bundle on the product $\mathfrak{M}_v \times \mathfrak{M}_w$. If one assumes further that the splitting $\mathcal{O}(\Theta) = \Theta_w \boxtimes \Theta_v$ holds (which is true, e.g., if one of the moduli spaces is simply connected), then Θ induces a linear map

$$SD : H^0(\mathfrak{M}_v, \Theta_w)^\vee \longrightarrow H^0(\mathfrak{M}_w, \Theta_v) \tag{2}$$

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called *Strange Duality* map. One can then ask the following natural questions: in all cases when one has that the equality in terms of dimension

$$h^0(\mathfrak{M}_v, \Theta_w) = h^0(\mathfrak{M}_w, \Theta_v)$$

is the map SD an isomorphism?

Strange Duality was first proved for curves, in [Bel08, MO07]. The corresponding version for surfaces saw some partial results when $X = \mathbb{P}^2$ by Danila [Dan04], and a generic version for K3 surfaces by Marian and Oprea in [MO13]. The abelian case is a rich story itself, and it will be the main focus of Part I.

0.1.1 The abelian case

In the case when X is an abelian surface, we denote by $\mathcal{M}_v$ the moduli space of Gieseker semistable sheaves of type v, by the pair $\mathcal{M}_v^+$ (respectively $\mathcal{M}_v^-$) is the moduli space of semistable sheaves on X of type v with *fixed determinant* (respectively, whose image via the Fourier-Mukai transforms with kernel the Poincaré bundle $\mathcal{P}$ have fixed determinant); and by K_v the *generalized Kummer variety*, i.e. the moduli space of semistable sheaves of type v whose determinant and whose determinant of the Fourier-Mukai transform are both fixed. It was shown [MO14b] that the Verlinde numbers are the same on three different pairs of moduli spaces: $(\mathcal{M}_v^+, \mathcal{M}_w^+)$, $(\mathcal{M}_v^-, \mathcal{M}_w^-)$, and $(K_v, \mathcal{M}_w)$. The first two cases were examined in [MO14a]. The third case is especially interesting because it arises from the unique geometry of abelian surfaces, and because of the apparent asymmetry in the choice of the two moduli spaces. In our joint paper [BMOY14] we prove that strange duality holds in the third case for a generic abelian surfaces. More precisely, one of the main results of our paper is the following:

Theorem 3 ([BMOY14], Theorem 2)**.** Assume (X, H) is a generic primitively polarized abelian surface, and v, w are two orthogonal Mukai vectors of ranks $r, r' \geq 2$ with

1. $c_1(v) = c_1(w) = H$;

2. $\chi(v) < 0, \chi(w) < 0$.

Then, the locus

$$\Theta = \{(E, F) \text{ with } \mathbb{H}^0(E \otimes^{\mathbf{L}} F) = 0\} \subset K_v \times \mathcal{M}_w$$

is a divisor, and induces an isomorphism

$$SD : H^0(K_v, \theta_w)^\vee \longrightarrow H^0(\mathcal{M}_w, \theta_v).$$

Our strategy to prove Theorem 3.5, and the subsequent organization of Part I, will be the following:

- In Chapter 3, we prove the statement in the rank one case. More precisely, we prove the following:

  Theorem 1. *Let $L \to X$ be an ample line bundle on an arbitrary abelian surface. Write $\chi(X, L) = \chi = a + b$ for positive integers a and b. The divisor*

  $$\Theta_L = \{(I_Z, I_W, y) \text{ with } H^0(I_Z \otimes I_W \otimes y \otimes L) \neq 0\} \subset K^{[a]} \times X^{[b]} \times \widehat{X}$$

  induces an isomorphism

  $$D_L : H^0(K^{[a]}, \Theta_v)^\vee \longrightarrow H^0(X^{[b]} \times \widehat{X}, \Theta_w).$$

  In other words, if we fix an ample line bundle $L \in \mathrm{Pic}(X)$ with $\chi(L) = a + b$ for positive integers a and b, we can set $v = (1, 0, -a)$ and $w = (1, c_1(L), a)$. Then we have the following isomorphisms:

  $$K_v \cong K^{[a]}, \quad \mathcal{M}_w \cong X^{[b]} \times \widehat{X}$$

  where $X^{[b]}$ is the Hilbert scheme of b points on X, $K^{[a]}$ denotes the generalized Kummer variety of a points adding to zero on X and $\widehat{X}$ is dual abelian surface of X. In this basic case, the Θ-divisor described above becomes the locus

$$\Theta_L = \{(I_z, I_w, y) \mid H^0(I_Z \otimes I_w \otimes y \otimes L) \neq 0\} \subset K^{[a]} \times X^{[b]} \times \widehat{X},$$

and the associated line bundle takes a very explicit form. In order to prove the strange duality conjecture in this setting, we construct another line bundle $\widehat{M^a}$, where M is some twist of L and $\widehat{M}$ denotes its Fourier-Mukai transform with kernel the Poincaré bundle and then we use the representation theory of the Heisenberg group $\mathcal{G}(\widehat{M^a})$, introduced by Mumford in his study of Abelian varieties (see [Mum66]) to prove the injectivity of the strange duality map. The surjectivity will then follow from a previous dimensional computation done in [MO14a].

- In Chapter 4, we use Fourier-Mukai techniques to prove the strange duality conjecture for any product abelian surface. More precisely, we prove the following:

  Theorem 2. *Let $X = B \times F$ be a product abelian surface. Assume v and w are two orthogonal Mukai vectors of ranks $r, r' \geq 2$ with*

  $$c_1(v) \cdot f = c_1(w) \cdot f = 1.$$

  Then, the locus

  $$\Theta = \{(E, F) \text{ with } \mathbb{H}^0(E \otimes^{\mathbf{L}} F) \neq 0\} \subset \mathcal{K}_v \times \mathcal{M}_w$$

  is a divisor, and induces an isomorphism

  $$\mathsf{D} : H^0(\mathcal{K}_v, \Theta_w)^\vee \to H^0(\mathcal{M}_w, \Theta_v).$$

  Recall that, given any two smooth projective varieties X and Y and a complex $\mathcal{E}$ in the bounded derived category $\mathcal{D}^b(X \times Y)$ the *Fourier-Mukai transform* from X to Y with kernel $\mathcal{E}$ is the functor

  $$\Phi_{\mathcal{E}} : \mathcal{D}^b(X) \to \mathcal{D}^b(Y), \ F \mapsto \mathbf{R}p_!(\mathcal{E} \otimes^{\mathbf{L}} q^*F),$$

where q and p are the projections to X and Y respectively. Historically, the first Fourier-Mukai transform was introduced by Mukai in the setting of abelian varieties: specifically, the source variety X is an abelian variety and the target variety $Y = \widehat{X}$ is its dual. Moreover, the kernel considered by Mukai is the *normalized Poincaré bundle*, i.e. the unique line bundle $\mathcal{P}$ on $X \times \widehat{X}$ such that

$$\mathcal{P}|_{\{x\} \times \widehat{X}} \cong \mathcal{O}_{\widehat{X}}, \ \mathcal{P}|_{X \times \{y\}} \cong y.$$

This specific Fourier-Mukai transform is fundamental in the theory of abelian varieties, and will be widely used in this exposition. It is an open conjecture whether the existence of a Fourier-Mukai transform which is an equivalence of categories gives rise to a birational isomorphisms between the two varieties. In our case, given a product abelian surface $X = B \times F$ with a suitable polarization, where B and F are elliptic curves, we use a fiberwise Fourier-Mukai transform which in fact gives rise to birational isomorphisms

$$K_v \dashrightarrow K^{[d_v]}, \ \mathcal{M}_w \dashrightarrow X^{[d_w]} \times \widehat{X}.$$

Here v and w are Mukai vectors with fiber degree one, and

$$d_v = \frac{1}{2} \dim \mathcal{M}_v - 1.$$

These birational isomorphisms turn out to be regular in codimension one, allowing us to use the result we obtained in the rank one case and prove the strange duality isomorphism in this new setting.

- In Chapter 5, we use a degeneration argument to prove the strange duality conjecture for generic abelian surfaces and we prove Theorem 3. We work in families on the

moduli stack of polarized abelian surfaces, where the polarization has fixed degree. The relative version of the spaces of generalized theta functions we are interested in (2) are the so-called *Verlinde sheaves*, whose properties were previously investigated by Marian and Oprea, see [MO14b]. Crucial to our argument is the fact that the Verlinde sheaves are generically vector bundles of equal rank, whose fibers are the the spaces of generalized theta functions. We use and prove the claim that the moduli spaces of sheaves we obtain by varying the Gieseker stability condition agree in codimension one each time a wall is crossed, allowing us to exchange the suitable polarization with the one imposed by the first Chern class of our Mukai vector when we move on the moduli stack of polarized abelian varieties. Our result then follows.

- In Chapter 6, we finally prove the *global* local freeness of the Verlinde sheaves. More precisely, we prove the following:

Theorem 3. *Let (X, H) be a polarized abelian surface. Assume that*

$$v = (r, dH, \chi), \ w = (r', d'H, \chi')$$

are orthogonal primitive Mukai vectors of ranks $r, r' \geq 2$ such that

(i) $d, d' > 0$;

(ii) $\chi < 0, \ \chi' < 0$.

Assume furthermore that if $(d, \chi) = (1, -1)$, then (X, H) is not a product of two elliptic curves. We have

$$h^0(\mathcal{K}_v, \Theta_w) = \chi(\mathcal{K}_v, \Theta_w) = \frac{d_v^2}{d_v + d_w} \binom{d_v + d_w}{d_v}.$$

Moreover, for any representative $F \in \mathcal{K}_w$,

$$h^0(\mathcal{M}_v, \Theta_F) = \chi(\mathcal{M}_v, \Theta_F) = \frac{d_w^2}{d_v + d_w} \binom{d_v + d_w}{d_v}.$$

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This is done by using results on the birational geometry of Bridgeland moduli spaces previously obtained by Bayer-Macrí and Minamide-Yanagida-Yoshioka, respectively in [BM14a], [MYY11]. This will be useful for future studies on strange duality.

0.1.2 Historical background on the SD morphism and the Verlinde numbers

It is significant to point out that the Euler characteristics of the Theta divisors, i.e. the numbers $\chi(\mathfrak{M}_v, \Theta_w)$, are of great importance in mathematical physics and are often referred to as *Verlinde numbers*. Historically, Verlinde numbers arise while doing geometric quantization in mathematical physics. Indeed, given a Riemann surface Σ of genus g, one can look at the moduli space $\mathscr{A}_0$ of flat differentiable connections on the trivial bundle $\Sigma \times \mathrm{SU}(r) \longrightarrow \Sigma$ modulo its group of gauge transformations $\mathscr{G} \cong \mathscr{C}^\infty(\Sigma, \mathrm{SU}(r))$. A classical theorem by Narasimhan and Seshadri [NS65] claims that such moduli space can be interpreted as a moduli space of semistable bundles on Σ. More precisely, one has the following identification:

$$\mathscr{A}_0/\mathscr{G} \cong \mathcal{SU}_r(\Sigma) := \text{moduli space of semistable holomorphic rank r}$$

$$\text{vector bundles on } \Sigma \text{ with trivial determinant.}$$

Such moduli space also has a natural symplectic structure induced by the Killing form on the Lie algebra $\mathfrak{su}(r)$. The problem of (pre-)quantizing[1] the moduli space $\mathscr{A}_0/\mathscr{G}$ as a symplectic manifold is then equivalent to quantizing the well known moduli space $\mathcal{SU}_r(\Sigma)$. It turns out that any power of the Theta line bundle is a pre-quantization bundle on the moduli space

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[1] i.e. associating to it a triple $(\mathcal{L}, \nabla, F)$ of a holomorphic line bundle with a connection whose curvature is a multiple of the symplectic form and a complex distribution on the tangent bundle of the manifold fulfilling some compatibility conditions

$\mathcal{SU}_r(\Sigma)$. Thus the space of sections of the divisor

$$\Theta := \left\{ [E] \in \mathcal{SU}_r(\Sigma) \; : \; \dim_{\mathbb{C}} H^0(\Sigma, E) > 0 \right\} \tag{3}$$

which is called space of *level k non-abelian Theta functions* is the *quantum Hilbert space* of the moduli space $\mathscr{A}_0/\mathscr{G}$. The dimension of this vector space is therefore of great interest, and it was computed in the following:

Theorem 0.1.1 ([Ver88] (Verlinde formula))**.** *Let*

$$z_k^{SU(r)}(g) := \dim_{\mathbb{C}} H^0(\mathcal{SU}_r(\Sigma)_J, \Theta^k), \; where \; g = g(\Sigma).$$

Then

$$z_k^{SU(r)}(g) = \left(\frac{r}{k+r} \right)^g \sum_{\substack{S \subset \{1,\dots,k+r\} \\ |S|=r}} \prod_{\substack{s \in S, t \notin S \\ 1 \leq r \leq k+r}} \left| 2 \sin \pi \frac{s-t}{r+k} \right|^{g-1}. \tag{4}$$

Among other reasons, the Verlinde formula is remarkable because:

- the expression on the right actually defines a natural number,

- it is polynomial in k, and

- the dimension does not depend on a specific complex structure we fix on Σ.

One has interest in computing the Verlinde numbers on moduli spaces of sheaves over surfaces. The first instance of it came in [EGL01], where the three authors computed the Verlinde formula on Hilbert schemes of points over arbitrary surfaces for some Theta divisors. Let us recall that if X is a (say simply connected) surface, the Hilbert scheme $X^{[n]}$ of n points on X can be identified with the moduli space of semistable sheaves of class $(1, 0, -n)$ on X. Hence any line bundle on X with class $(1, c_1(L), n)$ induces a Theta line bundle, usually denoted by $L^{[n]}$ on the Hilbert scheme. It can be furthermore shown that if L has no higher cohomology, neither does $L^{[n]}$ (cf. e.g. [Sca07]). Ellingsrud, Göttsche and Lehn showed that:

$$h^0(X^{[n]}, L^{[n]}) = \binom{h^0(L)}{n} = \binom{\chi(L)}{n}.$$

Afterwards, Göttsche, Nakajima and Yoshioka [GNY08] computed the Verlinde numbers on K3 surfaces using a deformation-theoretic argument and the hyperkähler geometry of the situation. Later, Marian and Oprea derived the formula for abelian surfaces in [MO14a].

The question whether the Strange Duality morphism is always an isomorphism in the surface case brings a high impact contribution to the study of linear series of divisors on moduli spaces of sheaves. It asks whether there is a duality between certain spaces of sections of Theta divisors, which will imply a duality between the corresponding linear series on the moduli spaces. The (conjectured) isomorphism goes under the name of *Strange Duality* morphism. In the curve case the topological type of a vector bundle can be easily seen to depend only on its rank and its determinant. We consider the two moduli spaces $\mathcal{U}(r, d)$ and $\mathcal{U}(r, \Lambda)$, parametrizing vector bundles of rank r and, respectively, degree d and determinant Λ. The well understood structure of the Picard group of both moduli spaces [Dre87] reveals that the line bundle associated to the jumping locus

$$\Theta_{r,F} = \{E \in \mathcal{U}(r, d) \ : \ H^0(E \otimes F) \neq 0\}$$

depends only on the rank and the determinant of F on $\mathcal{U}(r, d)$, and it depends only on the rank and the degree of F when restricted to $\mathcal{U}(r, \Lambda)$. In fact, the Picard group of the latter has a unique ample generator θ_r, so we have:

$$\Theta_{r,F} = \theta_r^l$$

for some l. When the fixed determinant $\Lambda = \mathcal{O}$, the moduli spaces appearing in the strange duality morphism are $\mathcal{SU}(r)$ and $\mathcal{U}(k, k(g-1))$, and the morphism itself becomes:

$$H^0(\mathcal{SU}(r), \theta_r^k)^\vee \longrightarrow H^0(\mathcal{U}(k, k(g-1)), \Theta_k^r)$$

where

$$\Theta_k = \{F \in \mathcal{U}(k, k(g-1)) \ : \ h^0(F) = h^1(F) \neq 0\}$$

is a naturally defined line bundle. When moreover we set $k = 1$, the Verlinde formula (4) simplifies dramatically:

$$z_1^{\mathrm{SU}(r)}(g) = \frac{r^g}{(r+1)^{g-1}} \prod_{p=1}^{r} \left| 2\sin\frac{\pi}{r+1} \right|^{g-1} = r^g$$

and the corresponding isomorphism

$$H^0(\mathcal{SU}(r), \theta_r)^\vee \longrightarrow H^0(\mathrm{Jac}^{g-1}(X), \Theta_1^r)$$

was established long before the Verlinde formula was known, using representation-theoretic techniques which will be generalized in the abelian case. Strange Duality was first proved generically, by Belkale in [Bel08], then a global statement was given Marian and Oprea in [MO07] , and afterwards it was proved using different techniques again by Belkale in a subsequent paper.

0.2 Bridgeland moduli spaces and wall crossing

0.2.1 MMP for moduli spaces of sheaves over a surface

Since the moment when moduli spaces became objects of interest in algebraic geometry, one of the main questions that were asked was how to classify all their minimal models. The first steps in this direction were moved by Hacon,McKernan,Cascini and Birkhar [BCHM10],

which lead to a better understanding of how to classify the minimal models of the moduli space of stable curves ([HH09], [HH13]) or the Kontsevich moduli spaces of stable maps ([CC10, CC11]). In these cases, we learned several different approaches:

- Run the MMP;

- Vary the moduli functor;

- Vary the GIT stability condition that was used to construct the moduli space.

When the moduli space we consider is the Hilbert scheme of points on a given surface, however, it is not clear how to deform the moduli functor at all. A solution is given by considering the Hilbert scheme as a moduli space of sheaves of some specific fixed type, and then by modifying the polarization used to construct the Gieseker stability condition. This, however, turns out not to cover all the minimal models for the Hilbert scheme. Recently, progress was made by Arcara, Bertram, Coskun and Huizenga [ABCH12] in the case of $\mathbb{P}^{2[n]}$, and an answer was provided by the introduction of a broader notion of stability conditions, due to Bridgeland in [Bri07], which allowed to construct moduli spaces of more complex objects, such as complex of sheaves. This idea was later picked up by Yoshioka [Yos12a] and later by Bayer and Macrí [BM14a], who generalized such technique to the moduli spaces of sheaves over abelian surfaces (respectively, K3 surfaces), providing also some important insight on how to relate the understanding of stability conditions on a surface to the understanding of the Nef and ample cones of the moduli spaces of Bridgeland stable objects and to the MMP for moduli spaces of sheaves. A detailed description of the Nef and Ample cones is showed to be achievable by constructing certain nef divisor classes arising in a natural way from Bridgeland stability condition, and is crucial to classifying the minimal models of the moduli spaces of sheaves. Since then, there has been a constant stream of results towards a better understanding of the MMP for Hilbert schemes of points on other kinds of surfaces, see e.g. [Nue14a] in the case of Enriques surfaces. One of the most interesting examples to

consider is that of general type surfaces: following the same path in this direction proves often too hard, given the general bad behavior of their moduli spaces of sheaves.

0.2.2 Smooth projective surfaces of irregularity zero

In our joint paper [BHL$^+$15], we look at smooth projective surfaces of irregularity zero, which include families of general type surfaces and Del Mezzo surfaces of degree one. Given such a surface X, we give criteria to study the ample cone of the Hilbert scheme of points $X^{[n]}$. In particular, we prove the following:

Theorem 0.2.1 ([BHL$^+$15], Theorem 1.2). *Let X be a smooth projective surface of irregularity 0, and suppose $\mathrm{Pic}X \cong \mathbb{Z}H$, where H is an ample divisor. Let $a > 0$ be the smallest integer such that aH is effective. If*

$$n \geq \max\{a^2H^2, p_a(aH) + 1\},$$

then Nef($X^{[n]}$) is spanned by the divisor $H^{[n]}$ and the divisor

$$\frac{1}{2}K_X^{[n]} + \left(\frac{a}{2} + \frac{n}{aH^2}\right)H^{[n]} - \frac{1}{2}B. \tag{$*$}$$

An orthogonal curve class is given by letting n points move in a g_n^1 on a curve in X of class aH.

In some specific examples, it is then possible to give a better bound in terms of the length n:

Theorem 0.2.2 ([BHL$^+$15], Theorem 1.3). *Let X be one of the following surfaces:*

1. a very general hypersurface in $\mathbb{P}^3$ of degree $d \geq 4$, or

2. a very general degree d cyclic branched cover of $\mathbb{P}^2$ of general type.

In either case, $\mathrm{Pic}(X) \cong \mathbb{Z}H$ with H effective. Suppose $n \geq d-1$ in the first case, and $n \geq d$ in the second case. Then Nef($X^{[n]}$) is spanned by $H^{[n]}$ and the divisor class $(*)$ with $a = 1$.

Our strategy will be as follows:

- in Chapter 9, we will give a description of the higher rank walls.

- in Chapter 10, we will focus on surfaces with Picard rank one. Under this hypotesis, the Nef cones is two-dimensional, hence it is sufficient to find two extremal rays in order to have a complete description of it.

- in Chapter 11, we will further restrict to the case of Del Pezzo surfaces of degree one.

Part I

Part I: Generic Strange Duality on abelian surfaces

Chapter 1

Preliminaries

1.1 Generalities on hyperkähler manifolds

A compact Kähler surface X is a K3 surface if it is simply connected and it carries a global homolorphic symplectic form (i.e. the canonical bundle $K_X \cong \mathcal{O}_X$). An example is given by the *Fermat quartic*: consider the degree four polynomial $P(X_0, ..., X_3) = X_0^4 + X_1^4 + X_2^4 + X_3^4 \in \mathbb{C}[X_0, ..., X_3]$. The vanishing locus $S = V(P)$ is an irreducible quartic hypersurface in $\mathbb{P}^3_\mathbb{C}$, which is simply connected by the Lefschetz Hyperplane Theorem, and has canonical bundle $K_S = (\mathcal{O}_{\mathbb{P}^3}(-4) \otimes \mathcal{O}_{\mathbb{P}^3}(4))|_S \cong \mathcal{O}_S$ by adjunction. Hence, the surface S is a K3 surface and, by applying the same reasoning verbatim, every irreducible quartic hypersurface in $\mathbb{P}^3_\mathbb{C}$ is. K3 surfaces play a fundamental role in the classification of algebraic surfaces, hence it is natural to look for generalizations in higher dimensions. The following (beautiful) classification theorem motivates the definition of a hyperkähler manifold (HK):

Theorem 1.1.1 (Beauville-Bogomolov decomposition, [Bea83])**.** *Let X be a compact Kähler manifold with $c_1(X) = 0$. There exists an étale finite cover $\prod_{i=1}^d M_i \longrightarrow X$ where each of the factors M_i is either a compact complex torus, a Calabi-Yau variety or a HK.*

A *Calabi-Yau* variety (CY) is a compact Kähler manifold M of dimension $n \geq 3$ with trivial

canonical bundle and such that the Hodge numbers $h^{p,0}(X)$ vanish for $0 < p < n$. These can very well be considered generalizations of K3 surfaces, which indeed satisfy $h^{1,0}(X) = 0$.[1] Nonetheless, most of the strikingly interesting properties of K3 surfaces come from the existence of a symplectic structure which is compatible with the complex holomorphic structure, hence a more satisfactory generalization can be found in the following definition:

Definition 1.1.2. A compact Kähler manifold X is *hyperkähler* if it is simply connected and the space of its global holomorphic two-forms is spanned by a symplectic form.

The only varieties which are both CY and HK are K3 surfaces, since the vanishing condition $h^{p,0}(X) = 0$ for $0 < p < \dim X$ is compatible with the existence of a global holomorphic symplectic form only when $\dim X = 2$. In higher dimension, the first two families of examples were produced by Beauville in [Bea83]: the Hilbert scheme of points over a K3 surface, and a suitable subvariety of the Hilbert scheme of points over an abelian surface, called *generalized Kummer variety*. More families of examples have been constructed since then, but they can all be shown to be deformation equivalent to one of the two families already found by Beauville. Recently, two more sporadic examples were found by O'Grady in [O'G99] and [O'G03] by desingularizing a singular moduli space of sheaves on a K3 (respectively, abelian) surface. The HK manifolds this obtained are ten (respectively, six) dimensional varieties, which we will denote by $\widetilde{\mathcal{M}}_{10}$ (respectively, $\widetilde{\mathcal{M}}_6$).

1.2 Compact hyperkähler manifolds

We will now discuss some of the most interesting properties of compact hyperkähler manifolds, some of which are direct generalizations of the main properties of K3 surfaces.

---

[1]Indeed, the first homology group $H_1(X, \mathbb{Z})$ is the abelianization of the first homotopy group $\pi_1(X)$, hence the simple connectedness of X implies the vanishing of the first Betti number $b_1(X)$. Now, since X is Kähler, one has that $b_1(X) = h^{1,0} + h^{0,1} = 2h^{1,0}$.

- **(Hyperkähler geometry)** The reason behind the name "hyperkähler" lies in the following fact: if X is a simply connected Kähler manifold with a global holomorphic symplectic form spanning the space $H^{2,0}(X)$, then by Yau's solution to Calabi's conjecture there exists a Riemannian metric g on X such that the holonomy of (X, g) is isomorphich to the tautological representation of the symplectic group

$$Sp(r) := \{\phi : \mathbb{H}^r \longrightarrow \mathbb{H}^r \mid \phi \text{ is right-linear and } \overline{\phi(v)}^t \cdot \phi(w) = \bar{v}^t \cdot w\}$$

on $\mathbb{H}^r$, where $\dim X = 4r$. This can be interpreted as the existence of a quaternionic structure on X, meaning that there are three distinct complex structures I, J and K satisfying the quaternionic relations

$$I^2 = J^2 = K^2 = IJK = -1.$$

Conversely, any manifold (X, I, J, K), having a quaternionic structure as above can be thought as a complex manifold (X, I) with a global holomorphic symplectic form given by the fact that $Sp(r) = U(\mathbb{H}^r) \cap SO(\mathbb{H}^r)$. It can be also shown that $\pi_1(X) = 0$ (see e.g. [Bea83]), hence X is a HK manifold.

Remark 1.2.1. *Compact hyperkähler manifolds are often referred to as irreducible holomorphic symplectic (IHS), because of the existence of a global symplectic form spanning the space of homolorphic two-forms.*

There are many examples of non-compact hyperkähler manifolds (some of the most interesting both in representation theory being and in algebraic geometry the Nakajima quiver varieties), and some recent results show that the local structure of compact HK manifolds can be understood by means of those (see e.g. [AS15]).

- **(The Beauville-Bogomolov form)** The standard intersection pairing on the middle cohomology group $H^2(S) := H^2(S, \mathbb{C})$ of a K3 surface S can be shown to be even,

unimodular and of signature $(3, 19)$, hence isomorphic to the lattice

$$\Lambda = U^{\oplus 3} \oplus E_8(-1)^{\oplus 2}.$$

Similarly, if X is HK and ω is its symplectic form, we have the following:

Theorem 1.2.2 (Beauville [Bea83] and Fujiki [Fuj87]). *There exists a positive rational number c_X (Fujiki's constant) and an integral indivisible non-degenerate symmetric bilinear form $(,)_{BB}$ on $H^2(X)$ (Beauville-Bogomolov's form) of signature $(3, b_2(X)-3)$ such that the following hold:*

1. *$(\omega, \overline{\omega})_{BB} > 0$,*

2. *$\int_X \alpha^{2n} = c_X \cdot (\alpha, \alpha)_{BB}^n$ for $\alpha \in H^2(X)$,*

3. *$(\alpha, \alpha')_{BB} = 0$ if $\alpha \in H^{p,2-p}(X)$, $\alpha' \in H^{p',2-p'}(X)$ with $p + p' \neq 2$.*

The Beauville-Bolomolov form endows the cohomology group $H^2(X)$ with the structure of a lattice. Such lattice structure, and the respective Fujiki constant, can be explicitly computed for all the known examples (see e.g. [Rap08]).

- **(Birational projective HK)** A well-known fact about K3 surfaces is that they are all deformation equivalent to each others. This was first proved by Kodaira in [Kod64], and it is a consequence of the fact that every regular surface with trivial canonical bundle can be deformed to the Fermat quartic. Moreover, if any two K3 are birational equivalent, they can be shown to be isomorphic: indeed, the minimal model of a surface of non-negative Kodaira dimension is unique. In higher dimension the situation changes, but the above statement can be replaced by the following:

  Theorem 1.2.3 (Essentially Theorem 4.6, [Huy99]). *Two birational projective irreducible symplectic manifolds are deformation equivalent and, hence, diffeomorphic.*

The result was used to show that most of the known examples, with the exception of O'Grady's exceptional examples in dimension six and ten, are deformations of the two standard series provided by Hilbert schemes of points on K3 surfaces and generalized Kummer varieties. It was first shown with projective techniques for projective HK (under additional assumptions), and later extended to the current form using the same techniques employed in the proof of the Torelli theorem for HK.

1.3 Examples: moduli spaces of sheaves on a K3 surface and generalized Kummer varieties

Let S be a surface. The *Mukai vector* $v(F)$ of a sheaf F on S is the cohomology class $v(F) = \mathrm{ch}(F)\sqrt{\mathrm{Td}_S}$, where Td_S is the *Todd class* of the surface S. If we write any two Mukai vectors in the form $v = (r, \Lambda, s)$ and $v' = (r', \Lambda', s')$ in $H^*_{alg}(X, \mathbb{Q}) = \mathbb{Z} \oplus \mathrm{NS}(X) \oplus \mathbb{Z}$, the *Mukai pairing* is the bilinear form

$$\langle v, v' \rangle = \int_S \Lambda \cdot \Lambda' - rs' - r's.$$

The Mukai vector, when seen as a morphism from the K-theory $K(S)$, endowed with the Euler pairing to the algebraic cohomology ring endowed with the Mukai pairing is an isometry, and a very convenient way to enclose the invariants of a sheaf. The main two examples we want to discuss are the following:

- **(The moduli space of sheaves over a K3 surface)** Let (X, H) be a polarized K3 surface, and v a Mukai vector. We denote by $\mathcal{M}_H(v)$ the moduli space of semistable sheaves on X with respect to H having Mukai vector v. The moduli space $\mathcal{M}_H(v)$ is a HK manifolds when the polarization H is generic with respect to v. It can be shown to be of dimension $\langle v, v \rangle + 2$ and deformation equivalent to the Hilbert scheme of points $X^{[n]}$, where $n = \frac{1}{2}\langle v, v \rangle + 1$.

- **(The generalized Kummer variety)** Let (X, H) be a polarized abelian surface, and denote by $X^{[n]}$ the Hilbert scheme of n points over X. one can consider the addition morphism $X^n \longrightarrow X$ sending an n-tuple $(x_1, ..., x_n) \mapsto x_1 + ... + x_n$. Such morphism is clearly invariant under the action of the symmetric group which permutes the factors of the product X^n, hence it descends to the symmetric product $X^{(n)}$. Let us denote by a the composition of the addition morphism $X^{(n)} \to X$ with the Hilbert-Chow morphism $X^{[n]} \to X^{(n)}$. The locus

$$K^{[n]} := \{Z = (x_1, ..., x_n) \in X^{[n]} \mid a(Z) = 0\}$$

is called *generalized Kummer variety*. It can be shown to be a compact HK manifold (see e.g. [Bea83]) of dimension $2(n-1)$ with second Betti number $b_2(K^{[n]}) = 7$. Similarly, denote by $\mathcal{M}_H(v)$ the moduli space of sheaves on X with Mukai vector v: one can consider the Albanese morphism

$$Alb : \mathcal{M}_H(v) \longrightarrow X \times \hat{X} \ , \ E \mapsto (\det E, \det \mathbf{R}\mathcal{S}(E)),$$

where $\mathbf{R}\mathcal{S}(E)$ denotes the Fourier-Mukai transform of E with respect to the Poincaré bundle (see next chapter). One can show that the fibers of the morphism Alb are all irreducible and isomorphic. Let us denote by

$$\mathcal{K}_v := Alb^{-1}(0, 0).$$

Then $\mathcal{K}_v$ is a HK manifold which is deformation equivalent to $K^{[n]}$, where $n = \frac{1}{2}\langle v, v \rangle + 1$. The generalized Kummer variety, and its irreducible holomorphic symplectic structure, will be studied extensively in Part I.

1.4   Fourier-Mukai transforms

Fourier-Mukai transforms are a powerful tool when dealing with derived categories of coherent sheaves over a variety. Functors that are of Fourier-Mukai type behave well with respect

to elementary functorial operation: they admit left and right adjoints, the composition of two Fourier-Mukai transforms is again of Fourier-Mukai type and, finally, there are explicit conditions which allow one to decide whether a Fourier-Mukai transform is an equivalence or not. There are two main results in the theory of Fourier-Mukai transforms: the first one, which is due to Orlov (Theorem 1.4.3), states that each exact quivalence between two derived category $\mathcal{D}(X)$ and $\mathcal{D}(Y)$, where X and Y are smooth projective varieties, is isomorphic to an equivalence of Fourier-Mukai type. The second one, which is a corollary of Theorem 1.4.3, gives a description of the group $\mathrm{Aut}\mathcal{D}$ in some particular cases. It states that if X is a smooth projective variety with ample canonical or anticanonical sheaf, then the group $\mathrm{Aut}\mathcal{D}(X)$ is generated by shifts, automorphisms of the variety and twist by line bundles (we will later explain what a twist functor is). Fiber-wise Fourier-Mukai transforms will be introduced later and used extensively in chapter 4.

Let us now explain some heuristic behind the Fourier-Mukai transforms. First, recall what the classical the Fourier transform is. It is something like this: given a function $f(x)$, the Fourier transform of f is the function $g(y) := \int f(x)e^{2\pi ixy}dx$.

Let us give a quick description of the Fourier-Mukai transform:

1. Given two varieties X and Y, and a sheaf $\mathcal{P}$ on $X \times Y$. The sheaf $\mathcal{P}$ sometimes is called the "integral kernel". Take a sheaf $\mathcal{F}$ on X. Think of $\mathcal{F}$ as being analogous to the function $f(x)$ in the classical situation. Think of $\mathcal{P}$ as being the analogous, in the classical situation, of some function of x and y.

2. Now pull back the sheaf along the projection $q : X \times Y \to X$. Think of the pullback $q^*\mathcal{F}$ as being the analogous of the function $f(x)$, and of $\mathcal{P}$ as being analogous to the function $e^{2\pi ixy}$.

3. Next, take the tensor product $q^*\mathcal{F} \otimes \mathcal{P}$. This is analogous to the function $f(x)e^{2\pi ixy}$.

4. Finally, push down $q^*\mathcal{F} \otimes \mathcal{P}$ along the projection $p : X \times Y \to Y$. The result is the Fourier-Mukai transform of $\mathcal{F}$ — it is $p_*(q^*\mathcal{F} \otimes \mathcal{P})$. This last pushforward step can be thought of as "integration along the fiber": here the fiber direction is the X direction. So in the classical situation it is $g(y) = \int f(x)e^{2\pi ixy}dx$, which is the Fourier transform of $f(x)$.

To make all of this rigorous, we have to deal with derived categories of coherent sheaves, not just coherent sheaves. In this context the main difficulty is the pushforward operation. As is well known, the pushforward of a coherent sheaf is not always coherent. But we can use the derived pushfoward instead, at the "price" of having to deal with derived categories.

When X is an abelian variety, Y is the dual abelian variety, and $\mathcal{P}$ is the Poincare line bundle on $X \times Y$, then the Fourier-Mukai transform gives an equivalence of the derived category of coherent sheaves on X with the derived category of coherent sheaves on Y. This was proved by Mukai. This is supposed to be analogous to the statement I made about the classical Fourier transform being invertible. In other words the Poincare line bundle is really supposed to be analogous to the function $e^{2\pi ixy}$. A more general choice of $\mathcal{P}$ corresponds to, in the classical situation, so-called integral transforms, i.e. transforms of Fourier type with a different kernel. They do not have, in general, all the good properties of the Fourier transform, but they can be nonetheless studied to provide examples of transforms between functions in L^p spaces. This is probably why $\mathcal{P}$ is called the integral kernel: to recall the kernel of the Fourier-transform. When X is an abelian variety, therefore, the analogies between the classical Fourier transform and the Fourier-Mukai transforms are stronger, and some of the properties which make the Fourier transform a powerful tool in analysis are analogously resembled in the algebraic geometric version. This topic is completely treated in [Muk81].

In this chapter, we will give the definition of the Fourier-Mukai transform, and we will its

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basic properties and give some interesting examples of how it can be applied.

1.4.1 Definition and first properties

Let X and Y be smooth projective varieties[2], and let

$$
\begin{array}{ccc}
 & X \times Y & \\
{}^{q}\swarrow & & \searrow^{p} \\
X & & Y
\end{array}
$$

be the projection on each of the two factors. To each object $\mathcal{P} \in \mathcal{D}(X \times Y)$, we can associate an exact functor of triangulated categories $\Phi_{\mathcal{P}} : \mathcal{D}(X) \longrightarrow \mathcal{D}(Y)$, which is defined as follows:

$$
\Phi_{\mathcal{P}} : \mathcal{D}(X) \longrightarrow \mathcal{D}(Y)
$$

$$
\mathcal{F}^{\bullet} \mapsto Rp_{*}(\mathcal{P} \overset{L}{\otimes} Lq^{*}\mathcal{F}^{\bullet}).
$$

Notation 1.4.1. *Now and later, we will write f_{*}, f^{*},$\otimes$, $\mathcal{H}om$, Hom respectively for Lf_{*}, Rf^{*}, $\overset{L}{\otimes}$, $R\mathcal{H}om$, RHom: there is no risk of confusion, as we will always work in the derived context.*

Remark 1.4.2. *Be careful! The two notations $\mathrm{Hom}_{\mathcal{D}(X)}(\bullet, \bullet)$ and $\mathrm{Hom}(\bullet, \bullet)$ refer to different objects.*

Let us notice some a basic consequence of the definition. The functor $\Phi_{\mathcal{P}}$ is called *Fourier-Mukai Transform of kernel $\mathcal{P}$*. Notice that it is always exact, because it is the composition of three exact functors, namely p_{*}, q^{*} and $\mathcal{P}\otimes$, which we have proved to be exact in the first chapter. The exactness of a functor in the triangulated context is really important, much more than in the abelian context: an exact functor of triangulated categories commutes with the shift, which means more or less that we can treat each complex in the derived category like a direct sum of sheaves (in Proposition **??** we will prove that any complex is

---

[2]By a variety we mean an integral separated scheme of finite type over an algebraically closed field

isomorphic to a direct sum of shifted sheaf). A triangulated functor which is not exact is, therefore, much less manageable: for this reason, we will focus our attention exclusively on exact functors all through the chapter.

Properties:

1. The identity

$$id : \mathcal{D}(X) \longrightarrow \mathcal{D}(X)$$

   is isomorphic to the Fourier-Mukai transform with kernel $\mathcal{O}_\Delta$, where $\Delta \subset X \times X$ is the diagonal .

2. If $f : X \longrightarrow Y$ is a morphism between algebraic varieties, then

$$f_* \cong \Phi_{\mathcal{O}_{\Gamma_f}} : \mathcal{D}(X) \longrightarrow \mathcal{D}(Y).$$

3. The shift functor $[1] : \mathcal{D}(X) \longrightarrow \mathcal{D}(X)$ is isomorphic to the FMT with kernel $\mathcal{O}_\Delta[1]$.

4. The composition of two arbitrary FMT is again a FMT. Let X, Y, Z be smooth projective varieties over a field k, as in the introduction to the Chapter. Consider objects $\mathcal{P} \in \mathcal{D}(X \times Y)$, $\mathcal{Q} \in \mathcal{D}(Y \times Z)$. Then define an object $\mathcal{R}$ in $\mathcal{D}(X \times Z)$ by the formula:

$$\mathcal{R} := \pi_{XZ_*}(\pi_{XY}^* \mathcal{P} \otimes \pi_{YZ}^* \mathcal{Q}),$$

   where

$$
\begin{array}{ccc}
 & X \times Y \times Z & \\
\pi_{XY} \swarrow & \pi_{XZ} \downarrow & \searrow \pi_{YZ} \\
X \times Y & X \times Z & Y \times Z.
\end{array}
$$

Then one has $\Phi_{\mathcal{Q}} \circ \Phi_{\mathcal{P}} \cong \Phi_{\mathcal{R}}$, as displayed below:

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$$\mathcal{D}(X) \xrightarrow{\Phi_{\mathcal{P}}} \mathcal{D}(Y) \xrightarrow{\Phi_{\mathcal{Q}}} \mathcal{D}(Z).$$

For the proof, simply look at this commutative diagram:
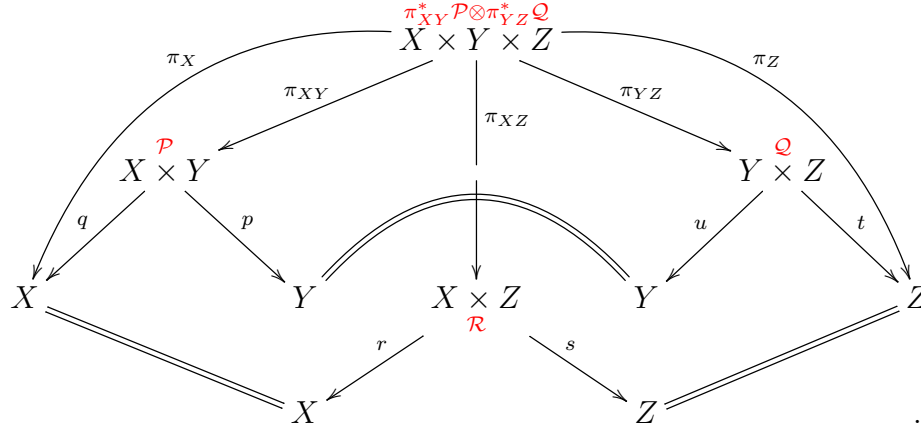
The importance of Fourier-Mukai transforms is shown in the following Theorem, which is due to Orlov.

Theorem 1.4.3. *Let X and Y be shooth projective varieties and let*

$$F : \mathcal{D}(X) \longrightarrow \mathcal{D}(Y)$$

be a fully faithful functor. If F admits left and tight adjoint functors, then there exists an object $\mathcal{P} \in \mathcal{D}(X \times Y)$ such that

$$\Phi_{\mathcal{P}} \cong F.$$

Proof. The proof is highly non-trivial, so we will just give references. There are two accounts of it in literature: the original one due to Orlov in [Orl02], and another one due to Kawamata in [Kaw02]. $\square$

The following example gives an idea of how one can lose information while passing from the objects in the derived category of the product to the corresponding Fourier-Mukai transforms.

1.4.2 Fourier-Mukai transforms on abelian varieties

Historically, the first Fourier-Mukai transform was introduced by Mukai in [Muk81] in the setting of abelian varieties: specifically, the source variety X is an abelian variety and the target variety $Y = \widehat{X}$ is its dual. Moreover, the kernel considered by Mukai is the *normalized Poincaré bundle*, i.e. the unique line bundle $\mathcal{P}$ on $X \times \widehat{X}$ such that

$$\mathcal{P}|_{\{x\} \times \widehat{X}} \cong \mathcal{O}_{\widehat{X}}, \ \mathcal{P}|_{X \times \{y\}} \cong y.$$

Definition 1.4.4. *The abelian FMT is the functor*

$$\mathbf{RS} : \mathcal{D}^b(X) \to \mathcal{D}^b(\widehat{X}) \ , \ \mathbf{RS}(\mathcal{F}^\bullet) = \mathbf{R}p_*(\mathcal{P} \otimes q^* \mathcal{F}^\bullet).$$

Mukai proved that such FMT is an equivalence. We would like to consider a few properties.

1. **(IT and WIT sheaves)** We say that WIT (weak index theorem) holds for a certain coherent sheaf F on X if $\mathbf{RS}(F)^i = 0$ for all but one i. In other words, we say that WIT holds if $\mathbf{RS}(F)$ is isomorphic to a complex concentrated in degree i, i.e. if $\mathbf{RS}(F) \cong \hat{F}[-i(F)]$ for some $i(F)$. We say, moreover, that IT holds for F if in addition the sheaf $\hat{F}$ is locally free.

   Example 1.4.5. *Any nondegenerate line bundle is IT_i for only one i and i is called the index of the line bundle.*

   We have the following result:

   Proposition 1.4.6. *[RBB09, Proposition 1.7, Corollary 3.4 and 3.5] Let $g := \dim X$.*

   (a) *A coherent sheaf F on X is IT_i if and only if $H^j(X, F \otimes \mathcal{P}_y) = 0$ for all $y \in Y$ and for all $j \neq i$, where $\mathcal{P}_y$ denotes the restriction of $\mathcal{P}$ to Xy. Furthermore, F is WIT_0 if and only if it is IT_0.*

   (b) *If F is WIT_i, then $\hat{F} := \mathbf{RS}(F)$ is WIT_{g-i}. Moreover $\hat{\hat{F}} \cong F$.*

(c) For every sheaf E on X, the sheaf $\mathbf{R}\mathcal{S}^0(E)$ is WIT_g, while the sheaf $\mathbf{R}\mathcal{S}^g(E)$ is WIT_0 (and hence IT_0 by part 1).

Skyscraper sheaves are IT_0, while the sheaves $\mathcal{P}_x$ are WIT_g but not IT, since it can be shown that for every $x \in X$ one has that $\widehat{\mathcal{P}}_x \cong \mathcal{O}_x[-g]$.

2. **(Cohomology of Poincaré bundle)**. It is possible to show that $\mathbf{R}\mathcal{S}(\mathcal{O}_X) \cong \mathcal{O}_{\hat{0}}[-g]$. This is equivalent to saying that:

$$R^i p_*(\mathcal{P}) = \begin{cases} \mathcal{O}_{\hat{0}} & \text{if } i = g \\ 0 & \text{otherwise.} \end{cases}$$

This immediately implies, by using a Leray spectral sequence:

$$H^i(X \times \widehat{X}, \mathcal{P}) = \begin{cases} \mathbb{C} & \text{if } i = g \\ 0 & \text{otherwise.} \end{cases}$$

3. **(Parseval's theorem)** Like many other analogies with the classical Fourier transform, a result that goes through is an analogue of Parseval's theorem. More specifically, the following holds:

Theorem 1.4.7. *[Muk81, Corollary 2.5] Assume that WIT holds for F and G. Then $\mathrm{Ext}^i_{\mathcal{O}_X}(F, G) \cong \mathrm{Ext}^{i+\mu}_{\mathcal{O}_{\hat{X}}}(\hat{F}, \hat{G})$ for every integer i, where $\mu = i(F) - i(G)$. Especially, we have an isomorphism $\mathrm{Ext}^i_{\mathcal{O}_X}(F, F) \cong \mathrm{Ext}^i_{\mathcal{O}_{\hat{X}}}(\hat{F}, \hat{F})$ for every i.*

This has many corollaries, but the one of the most interesting it is the following:

Corollary 1. *If F is an IT_i sheaf on X, then the Euler characteristic of F and the rank of F are related by*

$$\chi(X, F) = (-1)^i \mathrm{rk}(F).$$

Proof. One has $\chi(X, F) = \sum_j (-1)^j h^j(X, F) = \sum_j (-1)^j \dim \mathrm{Ext}^{j+g-i}(\mathcal{O}_{\hat{0}}, \hat{F})$. Since F is locally free, Serre duality gives $\mathrm{Ext}^{j+g-i}(\mathcal{O}_{\hat{0}}, \hat{F}) \cong h^0(X, \mathcal{O}_{\hat{0}} \otimes \hat{F})$. This vanishes unless $j = i$, and in that case we have $h^0(X, \mathcal{O}_{\hat{0}} \otimes \hat{F}) = \mathrm{rk}(F)$. $\square$

4. **(Cohomological FMT)** The topological invariants of the Abelian Fourier-Mukai transform can be computed by means of the Grothendieck-Riemann-Roch theorem in terms of the first Chern class of the Poincaré bundle. The cohomological FMT takes the form

$$\mathbf{R}\mathcal{S}^H(\alpha) = p_*(\mathrm{ch}(\mathcal{P}) \cdot q^*(\alpha))$$

for $\alpha \in H^*(X, \mathbb{Q})$. More explicitly, the Chern character of the Fourier-Mukai transform of a complex $\mathcal{F}^\bullet \in \mathcal{D}^b(X)$ is given by:

$$\mathrm{ch}_k(\mathbf{R}\mathcal{S}(\mathcal{F}^\bullet)) = \frac{1}{k!} p_*[c_1(\mathcal{P})^k \cdot \mathrm{ch}_{g-k}(\mathcal{F}^\bullet)] = \mathbf{R}\mathcal{S}^H(\mathrm{ch}_{g-k}(\mathcal{F}^\bullet)).$$

This really shows how the FMT "flips" the Chern character of a complex.

Let us now restrict to the case when $g = 1$, i.e. our abelian variety is an elliptic curve. There we have seen that, in view of property 3, the FMT with the standard Poincaré line budle as a kernel switches the rank and the degree (which on an elliptic curve is equal to the Euler characteristic by Riemann-Roch) of a sheaf. We now want to consider a slightly different setup. One can replace the target variety with the moduli space $Y := \mathcal{M}_X(a, b)$ of semistable vector bundles of rank a and degree b, for any pair of coprime integers a and b: by a result of Atiyah (see e.g. [Ati57, Tu93]) this is a fine moduli space, and there exists a canonical isomorphism

$$X \xrightarrow{\cong} \mathcal{M}_X(a, b) , \quad x \mapsto W_{a,b} \otimes \mathcal{O}_X(o_X - x)$$

where $W_{a,b}$ is the Atiyah bundle, i.e. the unique semistable vector bundle of rank a and determinant $\mathcal{O}_X(bo_X)$. Therefore, one can consider the universal family $\mathcal{P}_{a,b}$ over $X \times Y$,

which is going to satisfy the usual constraint:

$$\mathcal{P}_{a,b}|_{X\times\{E\}} \cong E$$

and let us denote by $n := \deg \mathcal{P}_{a,b}|_{\{x\}\times Y}$.

The cohomological FMT

$$\Phi^H_{\mathcal{P}_{a,b}} : \qquad H^*_{alg}(X,\mathbb{Z}) \qquad \rightarrow \qquad H^*_{alg}(Y,\mathbb{Z})$$
$$\|\qquad\qquad\qquad\qquad\|$$
$$\mathbb{Z}\oplus\mathbb{Z} \qquad\qquad\qquad \mathbb{Z}\oplus\mathbb{Z}$$

needs therefore to be representable by an integral, two by two matrix with determinant one (by property 3), hence an element of $SL_2(\mathbb{Z})$, say

$$M = \begin{pmatrix} x & y \\ z & w \end{pmatrix}.$$

Now, in order to find out the coefficients of such matrix, it is enough to compute:

$$\begin{pmatrix} x & y \\ z & w \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} x \\ z \end{pmatrix}$$

and

$$\begin{pmatrix} x & y \\ z & w \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} y \\ w \end{pmatrix},$$

that is,

$$(x,z) = \mathrm{ch}(\mathbf{R}p_*(\mathcal{P}_{a,b})) \text{ and } (y,w) = \mathrm{ch}(\mathbf{R}p_*(\mathcal{P}_{a,b} \otimes q^*\mathcal{O}_0)).$$

Since $\mathrm{rkNS}(X\times Y) = 3$ and $\mathrm{NS}(X\times Y)$ is generated by the divisor classes $\Delta, [X\times 0]$ and $[0\times Y]$, the class Δ being the graph of the natural isomorphism $X \cong Y$, one has that the first Chern class of the vector bundle $\mathcal{P}_{a,b}$ is of the form

$$c_1(\mathcal{P}_{a,b}) = s\Delta + t[X\times 0] + u[0\times Y]$$

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for some s, t and u integers. In fact, by restricting the vector bundle $\mathcal{P}_{a,b}$ to each factor, it is possible to show (cf. e.g. [MO14a]) that

$$c_1(\mathcal{P}_{a,b}) = \Delta + (n-1)[X \times 0] + (b-1)[0 \times Y],$$

Therefore che Chern character of $\mathcal{P}_{a,b}$ is

$$\mathrm{ch}(\mathcal{P}_{a,b}) = a + \Delta + (n-1)[X \times 0] + (b-1)[0 \times Y] + \chi\omega,$$

where $\chi = \chi(X \times Y, \mathcal{P}_{a,b})$ and ω is the class of a point. Hence we calculate:

$$
\begin{aligned}
\mathrm{ch}(\mathbf{R}p_*(\mathcal{P}_{a,b})) &= \mathbf{R}p_*\mathrm{ch}(\mathcal{P}_{a,b})) \\
&= \mathbf{R}p_*(a + \Delta + (n-1)[X \times 0] + (b-1)[0 \times Y] + \chi\omega) \\
&= b + \chi\omega,
\end{aligned}
$$

therefore $(x, z) = (b, \chi)$. Now we will need to calculate

$$(y, w) = (\mathrm{rk}\mathbf{R}p_*(\mathcal{P}_{a,b} \otimes q^*\mathcal{O}_0), \deg \mathbf{R}p_*(\mathcal{P}_{a,b} \otimes q^*\mathcal{O}_0)).$$

Cohomology and base change implies that, for any $E \in Y$:

$$
\begin{aligned}
\mathrm{rk}\mathbf{R}p_*(\mathcal{P}_{a,b} \otimes q^*\mathcal{O}_0) &= \chi(E \otimes \mathcal{O}_0) \\
&= h^0(E \otimes \mathcal{O}_0) \\
&= \mathrm{rk}E = a,
\end{aligned}
$$

while by Riemann-Roch we compute:

$$\deg \mathbf{R}p_*(\mathcal{P}_{a,b} \otimes q^*\mathcal{O}_0)) = \int_X c_1 \mathbf{R}p_*(\mathcal{P}_{a,b} \otimes q^*\mathcal{O}_0))$$
$$= \int_X [\mathrm{ch}\mathbf{R}p_*(\mathcal{P}_{a,b} \otimes q^*\mathcal{O}_0))]_1$$
$$= \int_X a + n\omega$$
$$= n.$$

Therefore, we find that:

$$M = \begin{pmatrix} b & a \\ \chi & n \end{pmatrix},$$

and $bn - a\chi = 1$. This also shows that a vector bundle with Chern character $n - \chi\omega$ is sent to a degree zero line bundle:

$$\begin{pmatrix} b & a \\ \chi & n \end{pmatrix} \begin{pmatrix} n \\ -\chi \end{pmatrix} = \begin{pmatrix} bn - a\chi \\ \chi n - n\chi \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

The FMT with this generalized kernel does not quite exchange rank and degree as the classical FMT does, but we have just shown that it is still possible to completely work out how Chern characters are transformed.

1.4.3 Fiber-wise Fourier-Mukai transforms

In this subsection, we consider a relative version of the Fourier-Mukai functors we studied in the previous subsection. This will allow us to work in the framework of schemes over a base. We consider proper morphisms of algebraic surfaces $X \to C$ and $Y \to C$, where C is a smooth algebraic curve, and use an element in the derived category of the fibered product $X \times_C Y$ as a kernel to define an integral functor between the derived categories of X and

Y. In the special case we are looking at, each of the two morphisms $X \to C$ and $Y \to C$ will be a *genus one fibration*, i.e. a surjective morphisms having a curve of arithmetic genus one as a general fiber. In this setup, we will be allowed to apply the standard Fourier-Mukai machinery we developed for abelian varieties in a relative fashion, giving us the possibility to extend such machinery to a larger class of algebraic varieties.

Consider proper morphisms of algebraic varieties $\pi_X : X \longrightarrow C$ and $\pi_Y : Y \longrightarrow C$. We denote by p and q the projections of the fiber product $X \times_C Y$ onto the first (respectively, the second) factor, and by $\pi = \pi_X \circ p = \pi_Y \circ q$ the projection onto the base curce C. We have the following cartesian diagram:

$$
\begin{array}{ccccc}
 & & X \times_C Y & & \\
 & {\scriptstyle p}\swarrow & \downarrow {\scriptstyle \pi} & \searrow {\scriptstyle q} & \\
X & & & & Y \\
 & {\scriptstyle \pi_X}\searrow & \downarrow & \swarrow {\scriptstyle \pi_Y} & \\
 & & C. & &
\end{array}
$$

For any object $\mathcal{E}^\bullet \in \mathcal{D}^b(\mathrm{Coh}(X \times_C Y))$, we can construct a relative Fourier-Mukai transform as follows:

$$
\Phi_{\mathcal{E}^\bullet} : \mathcal{D}^b(\mathrm{Coh}(X)) \to \mathcal{D}^b(\mathrm{Coh}(Y)) \ , \ \ \Phi_{\mathcal{E}^\bullet}(\mathcal{F}^\bullet) = \mathbf{R}q_*(\mathcal{E}^\bullet \otimes^L p^*\mathcal{F}^\bullet)
$$

Regarding the kernel $\mathcal{E}^\bullet$ as an object in $\mathcal{D}^b(\mathrm{Coh}(X \times Y))$ via the pushforward

$$
j_* : \mathcal{D}^b(\mathrm{Coh}(X \times_C Y)) \hookrightarrow \mathcal{D}^b(\mathrm{Coh}(X \times Y)),
$$

where $j : X \times_C Y \hookrightarrow X \times Y$ in the standard inclusion, and then taking the standard Fourier-Mukai transform would yield the same result.

Consider now the case where $\pi : X \to C$ is a relatively minimal genus one fibration with X an algebraic surface, and let f be the class of a smooth fiber. We denote by λ_X the minimal

fiber degree, i.e. the minimal positive integer such that the fibration $\pi : X \to C$ admits a holomorphic λ_X-multisection, and let σ be such multisection. Given a sheaf E on X, we denote its Chern character by the triple

$$\mathrm{ch}(E) = (r(E), c_1(E), c_2(E)) \in \mathbb{Z} \times \mathrm{NS}(X) \times \mathbb{Z}.$$

We are interested in looking at the *fiber degree* of E, i.e. at the integer

$$d(E) := c_1(E) \cdot f.$$

We are going to exploit the machinery we introduced in the previous subsection in a relative fashion. Consider invariants $v = (r, c_1, c_2)$ such that the coprimality condition

$$(r, d) = (r, c_1 \cdot f) = 1$$

is satisfied. Take also a suitable polarization in the sense of Friedman ([Fri98]), i.e. a polarization H such that

$$H = \sigma + Nf \ , \ N \gg 0$$

. One can form the moduli space $\mathcal{M}_H(v)$, as before. Since the polarization H reads stability in terms of stability on the fibers, we have a moduli space that looks like a family of moduli spaces over elliptic curves varying on a basis. One can then hope for a relative version of the result we had before. In particular, for each pair of coprime integers a and b, one can construct a relative compactified Jacobian $Y := \mathrm{Jac}_X(a, b)$, which can be seen either as the component of the relative moduli space of relatively stable sheaves $\mathcal{M}(X/C) \to C$ containing rank a, degree b sheaves, or as the moduli space of rank zero sheaves supported on fibers $\mathcal{M}_X(0, af, b)$. It is possible to prove that the compactified relative Jacobian is irreducible and projective, and it admits a natural genus one fibration $\hat{\pi} : \mathrm{Jac}_X(a, b) \to C$ itself. We have the following result:

Theorem 1.4.8. *[Bri97] The moduli space $\mathcal{M} = \mathcal{M}_H(r, c_1, c_2)$ is a smooth (non-empty) projective variety and is birationally equivalent to*

$$\mathrm{Pic}^0(\mathrm{Jac}_X(a, b)) \times \mathrm{Hilb}^t(\mathrm{Jac}_X(a, b)),$$

where (a, b) is the unique pair of integers satisfying $br - a(c_1 \cdot f) = 1$ and $0 < a < r$.

Furthermore, if $r >$ at the birational equivalence extends to give an isomorphism of varieties.

This is achieved by using a Fourier-Mukai transfom with the relative kernel $\mathcal{P}_{a,b}$ on the product $X \times_C \mathrm{Jac}_X(a, b)$: such kernel is a relarive universal family for the relative moduli space $\mathrm{Jac}_X(a, b)$, namely

$$\mathcal{P}_{a,b}|_{X_{\hat{\pi}(E)} \times \{E\}} \cong E$$

for every point $E \in (\mathrm{Jac}_X(a, b))$ representing a stable rank a, degree b vector bundle on the fiber $X_{\hat{\pi}}(E)$. It is possible to show that such a kernel gives an equivalence, which is also regular in codimension two, and which sends sheaves on X with invariants (r, c_1, c_2) to rank one sheaves on the relative Jacobian.

We are going to use this result extensively in chapter 4.

Chapter 2

The setup

This chapter appears in the paper [BMOY14] as Section 1.

Three versions of Le Potier's strange duality conjecture were formulated in [MO14a] for a polarized abelian surface (X, H). We recall them briefly.

For a sheaf $E \to X$, we write

$$v(E) = \operatorname{ch} E \in H^{2\star}(X, \mathbb{Z})$$

for its Mukai vector. For two Mukai vectors

$$v = (v_0, v_2, v_4), \quad w = (w_0, w_2, w_4) \in H^{2\star}(X, \mathbb{Z}),$$

the Mukai pairing is given by

$$\langle v, w \rangle = \int_X v_2 w_2 - v_0 w_4 - v_4 w_0.$$

We also set standardly

$$v^\vee = (v_0, -v_2, v_4) \in H^{2\star}(X, \mathbb{Z}).$$

Let $\mathcal{M}_v$ be the moduli space of Gieseker H-semistable sheaves with Mukai vector v. When v is primitive and the polarization H is generic, the moduli space $\mathcal{M}_v$ consists of stable

sheaves only, and is smooth projective of dimension

$$\dim \mathcal{M}_v = 2d_v + 2, \ \text{ where } d_v = \frac{1}{2}\langle v, v \rangle.$$

We will make this assumption about the moduli spaces throughout the paper, unless specified otherwise. We furthermore consider three subspaces of $\mathcal{M}_v$:

- the space $\mathcal{M}_v^+$ of sheaves with a fixed determinant line bundle;

- the space $\mathcal{M}_v^-$ of sheaves with fixed determinant of their Fourier-Mukai transform;

- the space $\mathcal{K}_v$ of sheaves for which both the determinant and the determinant of their Fourier-Mukai transform is fixed.

In introducing the spaces $\mathcal{M}_v^-$, $\mathcal{K}_v$, we use the Fourier-Mukai transform

$$\mathbf{R}\mathcal{S} : \mathbf{D}(X) \longrightarrow \mathbf{D}(\widehat{X})$$

with respect to the standardly normalized Poincaré line bundle

$$\mathcal{P} \to X \times \widehat{X}.$$

The moduli space $\mathcal{K}_v$ is precisely the fiber of the Albanese map

$$\mathsf{a} : \mathcal{M}_v \to X \times \widehat{X}.$$

The morphism a is defined up to the choice of a reference sheaf E_0 of type v. Specifically,

$$\mathsf{a}(E) = (\det \mathbf{R}\mathcal{S}(E) \otimes \det \mathbf{R}\mathcal{S}(E_0)^\vee, \det E \otimes \det E_0^\vee).$$

Consider now two Mukai vectors v and w, orthogonal in the sense that

$$\langle v^\vee, w \rangle = -\chi(X, v \cdot w) = 0.$$

A sheaf $F \to X$ with Mukai vector

$$w = \mathrm{ch}(F) \in H^{2\star}(X, \mathbb{Z})$$

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gives rise to a line bundle

$$\Theta_F \to \mathcal{M}_v$$

by the standard determinant construction described in [LP96], [Li96]. Specifically, if a universal family $\mathcal{E} \to \mathcal{M}_v \times X$ exists, we set

$$\Theta_F = \det \mathbf{R}p_!(\mathcal{E} \otimes q^\star F)^{-1} \to \mathcal{M}_v, \tag{2.1}$$

where p, q are the two projections. By restriction, one gets line bundles on each of the subspaces $\mathcal{M}_v^+, \mathcal{M}_v^-, \mathcal{K}_v$.

Within a fixed Mukai class w, for each of the four moduli spaces considered, the dependence of the determinant line bundle on F takes a particular form, as explained in [MO14a]:

- on $\mathcal{K}_v$, the line bundle $\Theta_F = \Theta_w$ depends only on the Mukai class w of F;

- on $\mathcal{M}_v^+$, the line bundle Θ_F is constant as long as the determinant of F is fixed;

- on $\mathcal{M}_v^-$, the line bundle Θ_F is constant as long the determinant of the Fourier-Mukai transform of F is fixed;

- on $\mathcal{M}_v$, the line bundle Θ_F is constant as long as F has both its determinant and its FM-transform determinant fixed.

Keeping these variations in mind, we write Θ_w for the determinant line bundle on each of the four moduli spaces, suitably understood. The distinctions above are further highlighted by the numerical equalities, cf. [MO08a]:

$$\chi(\mathcal{K}_v, \Theta_w) = \chi(\mathcal{M}_w, \Theta_v) = \frac{d_v^2}{d_v + d_w}\binom{d_v + d_w}{d_v}, \tag{2.2}$$

$$\chi(\mathcal{M}_v^+, \Theta_w) = \chi(\mathcal{M}_w^+, \Theta_v) = \frac{1}{2}\frac{c_1(v \otimes w)^2}{d_v + d_w}\binom{d_v + d_w}{d_v},$$

$$\chi(\mathcal{M}_v^-, \Theta_w) = \chi(\mathcal{M}_w^-, \Theta_v) = \frac{1}{2}\frac{c_1(\hat{v} \otimes \hat{w})^2}{d_v + d_w}\binom{d_v + d_w}{d_v}.$$

Here, $\hat{v}$ and $\hat{w}$ denote the cohomological Fourier-Mukai transforms of v and w.

Following Le Potier's original strange duality proposal [LP92], it was shown in [MO14a] that the Brill-Noether divisors

$$\Theta^+ = \{(E, F) \text{ with } \mathbb{H}^0(E \otimes^{\mathbf{L}} F) \neq 0\} \subset \mathcal{M}_v^+ \times \mathcal{M}_w^+$$

and

$$\Theta^- = \{(E, F) \text{ with } \mathbb{H}^0(E \otimes^{\mathbf{L}} F) \neq 0\} \subset \mathcal{M}_v^- \times \mathcal{M}_w^-$$

induce isomorphisms of spaces of sections

$$\mathsf{D}^+ : H^0(\mathcal{M}_v^+, \Theta_w)^{\vee} \longrightarrow H^0(\mathcal{M}_w^+, \Theta_v),$$

$$\mathsf{D}^- : H^0(\mathcal{M}_v^-, \Theta_w)^{\vee} \longrightarrow H^0(\mathcal{M}_w^-, \Theta_v),$$

for infinitely many Mukai vectors v and w and for an abelian surface (X, H) which is a product of elliptic curves.

In this paper we focus on the third possible geometry, associated with the divisor

$$\Theta = \{(E, F) \text{ with } \mathbb{H}^0(E \otimes^{\mathbf{L}} F) \neq 0\} \subset \mathcal{K}_v \times \mathcal{M}_w.$$

The current setting is particularly interesting since it exhibits the fixed versus unfixed determinant asymmetry also present for moduli spaces of bundles over curves [Bea95]. In this asymmetric setup, we establish the duality generically for a large class of Mukai vectors v and w, as captured in our main Theorem 6 below. We now explain the salient points of the argument and state the most important results along the way.

The starting point is the case when v and w are Mukai vectors of rank 1. For each integer $a > 0$, we let $X^{[a]}$ be the Hilbert scheme of a points on X, and let

$$K^{[a]} \subset X^{[a]}$$

be the generalized Kummer variety of a points adding to zero on X. When rank $v =$ rank $w = 1$, we have

$$\mathcal{K}_v \simeq K^{[a]}, \quad \mathcal{M}_w \simeq X^{[b]} \times \widehat{X},$$

for suitable a, b. In this setup, we prove

Theorem 4. *Let $L \to X$ be an ample line bundle on an arbitrary abelian surface. Write $\chi(X, L) = \chi = a + b$ for positive integers a and b. The divisor*

$$\Theta_L = \{(I_Z, I_W, y) \text{ with } H^0(I_Z \otimes I_W \otimes y \otimes L) \neq 0\} \subset K^{[a]} \times X^{[b]} \times \widehat{X}$$

induces an isomorphism

$$D_L : H^0(K^{[a]}, \Theta_v)^\vee \longrightarrow H^0(X^{[b]} \times \widehat{X}, \Theta_w).$$

The analogous isomorphism when both sides involve the Hilbert schemes $X^{[a]}$ and $X^{[b]}$ and the theta bundles over them was shown to hold for all surfaces in [MO08b]. By contrast, Theorem 4 is a subtler statement specific to abelian surfaces. Its proof requires new ideas and is obtained using the representation theory of the Heisenberg group.

Paralleling [MO14b] and [MO14a], the above result implies strange duality for product abelian surfaces via Fourier-Mukai techniques. Specifically, for moduli spaces of sheaves which are stable with respect to a suitable polarization in the sense of Friedman [Fri98], we show

Theorem 5. *Let $X = B \times F$ be a product abelian surface. Assume v and w are two orthogonal Mukai vectors of ranks $r, r' \geq 2$ with*

$$c_1(v) \cdot f = c_1(w) \cdot f = 1.$$

Then, the locus

$$\Theta = \{(E, F) \text{ with } \mathbb{H}^0(E \otimes^{\mathbf{L}} F) \neq 0\} \subset \mathcal{K}_v \times \mathcal{M}_w$$

is a divisor, and induces an isomorphism

$$\mathsf{D} : H^0(\mathcal{K}_v, \Theta_w)^\vee \to H^0(\mathcal{M}_w, \Theta_v).$$

In order to move from the product geometry of Theorem 5 to a generic abelian surface, we study the Verlinde sheaves

$$\mathbb{V}, \mathbb{W} \to \mathcal{A}.$$

These are defined in Section 5, and encode the spaces of generalized theta functions $H^0(\mathcal{K}_v, \Theta_w)$ and $H^0(\mathcal{M}_w, \Theta_v)$ respectively, as the pair (X, H) varies in its moduli space $\mathcal{A}$. We need to ensure that the Verlinde sheaves are generically locally free of expected rank given by the holomorphic Euler characteristics (2.2):

$$\text{rank } \mathbb{V} = \text{rank } \mathbb{W} = \frac{d_v^2}{d_v + d_w} \binom{d_v + d_w}{d_v}.$$

We establish this in our situation by showing that for surfaces of Néron-Severi rank 1 the theta line bundles are big and nef, and therefore carry no higher cohomology. This yields the following generic strange duality statement, which constitutes our main result.

Theorem 6. *Assume (X, H) is a generic primitively polarized abelian surface, and v, w are two orthogonal Mukai vectors of ranks r, $r' \geq 2$ with*

(i) $c_1(v) = c_1(w) = H$;

(ii) $\chi(v) < 0, \chi(w) < 0$.

Then, the locus

$$\Theta = \{(E, F) \text{ with } \mathbb{H}^0(E \otimes^{\mathbf{L}} F) \neq 0\} \subset \mathcal{K}_v \times \mathcal{M}_w$$

is a divisor, and induces an isomorphism

$$\mathsf{D} : H^0(\mathcal{K}_v, \Theta_w)^\vee \longrightarrow H^0(\mathcal{M}_w, \Theta_v).$$

While the statements of Theorems 5 and 6 mirror the $K3$ and abelian cases studied in [MO13] and [MO14a], different arguments are needed in the current *asymmetric* abelian setup. Several technical assumptions present in [MO13] and [MO14a] are in addition removed, yielding stronger results.

Finally, in Section 6 we show in great generality that the Verlinde sheaves

$$\mathbb{V}, \mathbb{W} \to \mathcal{A}$$

are in fact locally free over the entire moduli space $\mathcal{A}$ even though the higher cohomology of theta line bundles may not vanish. Specifically, this is implied by the following

Theorem 7. *Let (X, H) be a polarized abelian surface. Assume that*

$$v = (r, dH, \chi), \ w = (r', d'H, \chi')$$

are orthogonal primitive Mukai vectors of ranks $r, r' \geq 2$ such that

(i) $d, d' > 0$;

(ii) $\chi < 0, \ \chi' < 0$.

Assume furthermore that if $(d, \chi) = (1, -1)$, then (X, H) is not a product of two elliptic curves. We have

$$h^0(\mathcal{K}_v, \Theta_w) = \chi(\mathcal{K}_v, \Theta_w) = \frac{d_v^2}{d_v + d_w} \binom{d_v + d_w}{d_v}.$$

Moreover, for any representative $F \in \mathcal{K}_w$,

$$h^0(\mathcal{M}_v, \Theta_F) = \chi(\mathcal{M}_v, \Theta_F) = \frac{d_w^2}{d_v + d_w} \binom{d_v + d_w}{d_v}.$$

The proof uses Bridgeland stability conditions, and relies on recent results concerning wall-crossing as stability varies. As walls are crossed, the dimensions of the space of sections do not change. Crucially, we show that we can move away from the Gieseker chamber to a chamber for which the theta line bundles become big and nef. In order to control the wall-crossings and complete the argument, we make use of the explicit description of the movable cone of the moduli space recently obtained in [Yos12b]; see also [BM14a].

Chapter 3

The rank one case

This chapter appears in the paper [BMOY14] as Section 2.

3.1 Notation and preliminaries

We let X be an arbitrary abelian surface and consider two Mukai vectors v and w with

$$\operatorname{rank} v = \operatorname{rank} w = 1.$$

Specifically, letting $L \to X$ be an ample line bundle, and writing $\chi(L) = a + b$ for positive integers a and b, we set

$$v = (1,\, 0,\, -a), \quad w = (1,\, c_1(L), a).$$

We then have

$$\mathcal{K}_v \simeq K^{[a]}, \quad \mathcal{M}_w \simeq X^{[b]} \times \widehat{X},$$

and the strange duality divisor is

$$\Theta_L = \{(I_Z, I_W, y) \text{ with } H^0(I_Z \otimes I_W \otimes y \otimes L) \neq 0\} \subset K^{[a]} \times X^{[b]} \times \widehat{X}. \qquad (3.1)$$

Conforming to standard notation, we next set

$$L^{[a]} = \det \mathbf{R}p_\star \left(\mathcal{O}_{\mathcal{Z}} \otimes q^\star L\right) \quad \text{on} \ \ X^{[a]},$$

where $\mathcal{Z} \subset X^{[a]} \times X$ is the universal subscheme, and p, q are the projections to $X^{[a]}$ and X respectively. Throughout this section we also use

$$L^{[a]} \to K^{[a]}$$

to denote the restriction of the determinant line bundle to $K^{[a]} \subset X^{[a]}$.

The divisor

$$\Theta_L^+ = \{(I_Z, I_W) \text{ with } H^0(I_Z \otimes I_W \otimes L) \neq 0\} \subset X^{[a]} \times X^{[b]}, \tag{3.2}$$

with associated line bundle

$$\mathcal{O}(\Theta_L^+) = L^{[a]} \boxtimes L^{[b]} \text{ over } X^{[a]} \times X^{[b]}$$

induces an isomorphism

$$D_L^+ : H^0(X^{[a]}, L^{[a]})^\vee \longrightarrow H^0(X^{[b]}, L^{[b]}). \tag{3.3}$$

This constitutes the simplest instance of the strange duality phenomenon on surfaces; the isomorphism is described in [MO08b] and holds uniformly irrespective of the choice of surface. Relative to this standard rank one setup, the divisor Θ_L represents a twist specific to the abelian geometry. In particular, the associated line bundle takes the more complicated form

$$\mathcal{O}(\Theta_L) = L^{[a]} \boxtimes L^{[b]} \boxtimes \widehat{L} \otimes (\mathsf{a}, \mathrm{id})^\star \mathcal{P} \quad \text{on } K^{[a]} \times X^{[b]} \times \widehat{X}, \tag{3.4}$$

where $\mathcal{P} \to X \times \widehat{X}$ is the Poincaré line bundle, and

$$\mathsf{a} : X^{[b]} \to X$$

denotes the addition of points using the group law. We have also set

$$\widehat{L} = \det \mathbf{R}\mathcal{S}(L)^{-1} \text{ on } \widehat{X}.$$

Expression (3.4) is obtained by restricting to each factor and using Mumford's see-saw theorem; a detailed explanation is found in Example 1 of [Opr11]. Establishing that the induced map on the spaces of sections

$$D_L : H^0(K^{[a]}, L^{[a]})^\vee \longrightarrow H^0\left(X^{[b]} \times \widehat{X}, L^{[b]} \boxtimes \widehat{L} \otimes (\mathsf{a}, \mathrm{id})^\star \mathcal{P}\right) \tag{3.5}$$

is an isomorphism requires new ideas which we now describe.

3.2 Proof of Theorem 4

To begin, note that both sides of (3.5) have equal dimensions given by the Euler characteristics (2.2). For the left hand side, this follows from either Lemma 3 or Example 9 in [Opr11]: both show the vanishing of the higher cohomology of

$$L^{[a]} \to K^{[a]}$$

under the assumption that $L \to X$ is ample. For the right hand side, we can invoke Proposition 5 of Section 6 which applies to the current context as well. A direct argument is also possible making use of the étale pullbacks of the proof below.

We rephrase the statement of the theorem in two steps. To start, let

$$\varphi_L : X \longrightarrow \widehat{X}, \quad \varphi_L(x) = t_x^\star L \otimes L^{-1}$$

be the Mumford homomorphism; we also make use of $\varphi_{\widehat{L}} : \widehat{X} \longrightarrow X$. Consider now the diagram

$$
\begin{array}{c}
K^{[a]} \times X^{[b]} \times \widehat{X} \\
\Big\downarrow{\scriptstyle \widehat{\Phi}} \qquad \searrow{\scriptstyle \Psi} \\
K^{[a]} \times X^{[b]} \times X \xrightarrow{\ \Gamma\ } X^{[a]} \times X^{[b]} \\
\Big\downarrow{\scriptstyle \Phi} \\
K^{[a]} \times X^{[b]} \times \widehat{X}
\end{array}
$$

where

$$
\begin{aligned}
\Phi(I_Z,\, I_W,\, x) &= \left(I_Z,\, t_x^\star I_W,\, \varphi_L(x)\right), \\
\widehat{\Phi}(I_Z,\, I_W,\, y) &= \left(I_Z,\, I_W,\, \varphi_{\widehat{L}}(y)\right), \\
\Gamma(I_Z,\, I_W,\, x) &= \left(t_{-x}^\star I_Z,\, I_W\right), \\
\Psi(I_Z,\, I_W,\, y) &= \left(t_{-\varphi_{\widehat{L}}(y)}^\star I_Z,\, I_W\right) \implies \Psi = \Gamma \circ \widehat{\Phi}.
\end{aligned}
$$

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All four maps are étale:

- Φ and $\widehat{\Phi}$ have degree $\chi^2 = \chi(L)^2 = \chi(\widehat{L})^2$;

- Γ has degree a^4 since it can be viewed as quotienting by the group of a-torsion points on X;

- $\Psi = \Gamma \circ \widehat{\Phi}$ has degree $a^4 \chi^2$.

We now pull back the divisor $\Theta_L \subset K^{[a]} \times X^{[b]} \times \widehat{X}$ *twice*, first by Φ and then by $\widehat{\Phi}$.

Pullback under Φ

At the first stage, we obtain

$$
\begin{aligned}
\Phi^\star \Theta_L &= \{(I_Z, I_W, x) \text{ with } H^0(I_Z \otimes t_x^\star I_W \otimes \varphi_L(x) \otimes L) \neq 0\} \\
&= \{(I_Z, I_W, x) \text{ with } H^0(I_Z \otimes t_x^\star I_W \otimes t_x^\star L) \neq 0\} \\
&= \{(I_Z, I_W, x) \text{ with } H^0(t_{-x}^\star I_Z \otimes I_W \otimes L) \neq 0\} \\
&= \Gamma^\star \Theta_L^+.
\end{aligned}
$$

By contrast with expression (3.4), the line bundle associated with $\Phi^\star \Theta_L$ has the simpler form

$$
\mathcal{O}(\Phi^\star \Theta_L) = \mathcal{O}(\Gamma^\star \Theta_L^+) = \Gamma^\star(L^{[a]} \boxtimes L^{[b]}) = L^{[a]} \boxtimes L^{[b]} \boxtimes L^a \text{ on } K^{[a]} \times X^{[b]} \times X.
$$

The pullback divisor induces the map $\Phi^\star D_L$ for which the diagram

$$
\begin{array}{ccc}
D_L : \ H^0(K^{[a]}, L^{[a]})^\vee & \longrightarrow & H^0\left(X^{[b]} \times \widehat{X}, L^{[b]} \boxtimes \widehat{L} \otimes (\mathsf{a}, \mathrm{id})^\star \mathcal{P}\right) \\
\Big\| & & \Big\downarrow {\scriptstyle \Phi^\star} \\
\Phi^\star D_L : H^0(K^{[a]}, L^{[a]})^\vee & \longrightarrow & H^0\left(X^{[b]} \times X, L^{[b]} \boxtimes L^a\right)
\end{array}
$$

commutes. To show the original duality map D_L is injective (and thus by equality of dimensions an isomorphism), it suffices to show that the simpler $\Phi^\star D_L$ in the above diagram is injective.

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Pullback under $\widehat{\Phi}$

The second pullback, under $\widehat{\Phi}$, yields the divisor

$$\widetilde{\Theta}_L = \widehat{\Phi}^\star \Phi^\star \Theta_L$$

associated with the line bundle

$$\mathcal{O}(\widetilde{\Theta}_L) = \widehat{\Phi}^\star(L^{[a]} \boxtimes L^{[b]} \boxtimes L^a) = L^{[a]} \boxtimes L^{[b]} \boxtimes \varphi_{\widehat{L}}^\star L^a \ \text{ on } \ K^{[a]} \times X^{[b]} \times \widehat{X}.$$

Crucially, by our previous interpretation of $\Phi^\star \Theta_L$, we also have

$$\widetilde{\Theta}_L = \widehat{\Phi}^\star \Phi^\star \Theta_L = \widehat{\Phi}^\star \Gamma^\star \Theta_L^+ = \Psi^\star \Theta_L^+.$$

By the same argument as before, to show that the original duality map (3.5) is an isomorphism, it suffices to show that

Proposition 1. *The morphism $\widetilde{D}_L : H^0(K^{[a]}, L^{[a]})^\vee \longrightarrow H^0(X^{[b]} \times \widehat{X}, L^{[b]} \boxtimes \varphi_{\widehat{L}}^\star L^a)$ induced by $\widetilde{\Theta}_L$ is injective.*

Proof. We interpret the duality map representation-theoretically, using the theory of discrete Heisenberg groups. $\widetilde{D}_L$ is better suited for such an interpretation than the seemingly simpler morphism $\Phi^\star D_L$ obtained at the previous stage.

We have seen above that

$$\Psi^\star \mathcal{O}(\Theta_L^+) = \Psi^\star(L^{[a]} \boxtimes L^{[b]}) = L^{[a]} \boxtimes L^{[b]} \boxtimes \varphi_{\widehat{L}}^\star L^a.$$

Up to numerical equivalence on $\widehat{X}$, we have $\varphi_{\widehat{L}}^\star L = \widehat{L}^\chi$. Thus, there exists $y \in \widehat{X}$ such that

$$\varphi_{\widehat{L}}^\star L = t_y^\star \widehat{L}^\chi.$$

We define

$$M = L \otimes y$$

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and calculate

$$\widehat{M} := \det \mathbf{R}\mathcal{S}(M)^{-1} \implies \widehat{M} = t_y^\star \widehat{L} \implies \widehat{M}^{a\chi} = t_y^\star \widehat{L}^{a\chi} = \varphi_{\widehat{L}}^\star L^a.$$

Therefore,

$$\Psi^\star(L^{[a]} \boxtimes L^{[b]}) = L^{[a]} \boxtimes L^{[b]} \boxtimes \widehat{M}^{a\chi}.$$

We let $\mathsf{G}(\widehat{M}^a)$ be the Heisenberg group of the line bundle $\widehat{M}^a \to \widehat{X}$, sitting in an exact sequence

$$1 \to \mathbb{C}^\star \to \mathsf{G}(\widehat{M}^a) \to \mathsf{H}(\widehat{M}^a) \to 1,$$

where the quotient is the abelian group

$$\mathsf{H}(\widehat{M}^a) = \{y, \ t_y^\star \widehat{M}^a \simeq \widehat{M}^a\} \subset \widehat{X}.$$

For an introduction to Heisenberg group actions in the theory of abelian varieties we refer the reader to [Mum66], for instance.

Importantly, by construction, the étale morphism

$$\Psi : K^{[a]} \times X^{[b]} \times \widehat{X} \longrightarrow X^{[a]} \times X^{[b]}$$

can be viewed precisely as quotienting by the abelian group $\mathsf{H}(\widehat{M}^a)$. The latter acts on $K^{[a]} \times \widehat{X}$ via

$$\eta \cdot (I_Z, y) = (t_{\varphi_{\widehat{M}}(\eta)}^\star I_Z, y + \eta)$$

and trivially on $X^{[b]}$. Thus, as a pullback of $L^{[a]} \to X^{[a]}$ under the quotienting map Ψ, the line bundle

$$L^{[a]} \boxtimes \widehat{M}^{a\chi} \to K^{[a]} \times \widehat{X}$$

is $\mathsf{H}(\widehat{M}^a)$-equivariant, in other words it is $\mathsf{G}(\widehat{M}^a)$-equivariant, such that the center acts with weight 0. Independently, it is clear that the line bundle $\widehat{M}^{a\chi} \to \widehat{X}$ is $\mathsf{G}(\widehat{M}^a)$-equivariant, the center acting with weight χ. It follows that

$$L^{[a]} \to K^{[a]}$$

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is also $\mathsf{G}(\widehat{M^a})$-equivariant, so that the center acts with weight $-\chi$. The spaces of sections

$$H^0(\widehat{X}, \widehat{M^a\chi}) \text{ and } H^0(K^{[a]}, L^{[a]})$$

are in turn acted on with weights χ and $-\chi$ respectively, and furthermore, we can write

$$H^0(X^{[a]}, L^{[a]}) = \left(H^0(\widehat{X}, \widehat{M^a\chi}) \otimes H^0(K^{[a]}, L^{[a]}) \right)^{\mathsf{H}(\widehat{M^a})}.$$

Taking into account the long-known isomorphism

$$D_L^+ : H^0(X^{[a]}, L^{[a]})^\vee \longrightarrow H^0(X^{[b]}, L^{[b]})$$

of equation (3.3), we see that the dual of the linear map $\widetilde{D}_L$ is the natural

$$\widetilde{D}_L^\vee : \left(H^0(\widehat{X}, \widehat{M^a\chi}) \otimes H^0(K^{[a]}, L^{[a]}) \right)^{\mathsf{H}(\widehat{M^a})} \otimes H^0(\widehat{X}, \widehat{M^a\chi})^\vee \longrightarrow H^0(K^{[a]}, L^{[a]}),$$

which pairs the vector space $H^0(\widehat{X}, \widehat{M^a\chi})$ and its dual. To conclude the proposition, we show now that this map is surjective.

Let $\{\mathsf{S}_\alpha\}_{\alpha \in I}$ denote the irreducible representations of $\mathsf{G}(\widehat{M^a})$ with the center acting with weight $-\chi$. Decomposing into irreducibles, we write

$$H^0(K^{[a]}, L^{[a]}) = \bigoplus_\alpha \mathsf{S}_\alpha \otimes \mathbb{C}^{m_\alpha}, \quad H^0(\widehat{X}, \widehat{M^a\chi})^\vee = \bigoplus_\alpha \mathsf{S}_\alpha \otimes \mathbb{C}^{n_\alpha},$$

and the duality map $\widetilde{D}_L^\vee$ is

$$\widetilde{D}_L^\vee : \left(\bigoplus_\alpha (\mathbb{C}^{n_\alpha})^\vee \otimes \mathbb{C}^{m_\alpha} \right) \otimes \left(\bigoplus_\beta \mathsf{S}_\beta \otimes \mathbb{C}^{n_\beta} \right) \longrightarrow \bigoplus_\alpha \mathsf{S}_\alpha \otimes \mathbb{C}^{m_\alpha},$$

given explicitly by the natural pairing of the multiplicity spaces $(\mathbb{C}^{n_\alpha})^\vee$ and $\mathbb{C}^{n_\alpha}$.

We conclude $\widetilde{D}_L^\vee$ fails to be surjective only if there is an irreducible S_α which appears with nonzero multiplicity $m_\alpha \neq 0$ in $H^0(K^{[a]}, L^{[a]})$, but fails to appear in $H^0(\widehat{X}, \widehat{M^a\chi})$, so $n_\alpha = 0$. This is precluded by the following result, which in level 2 is Proposition 3.7 in [Iye]. This ends the proof of the proposition, and therefore of Theorem 4. $\qquad\square$

Lemma 3.2.1. *Let A be an abelian surface and $M \to A$ an ample line bundle. For any integer $k \geq 0$, all irreducible representations with central weight k of the Heisenberg group $\mathsf{G}(M)$ appear in the $\mathsf{G}(M)$-module $H^0(A, M^k)$ with nonzero multiplicity.*

For the benefit of the reader, we give the quick argument, which we lifted from [Iye]. Consider the natural homomorphism $\mathsf{G}(M) \to \mathsf{G}(M^k)$ and write

$$\mathsf{K} \cong \mathsf{G}(M)/\mu_k$$

for its image. Fix S a representation of the Heisenberg group $\mathsf{G}(M)$ of weight k. Certainly, S is a representation of K with weight 1. The induced representation

$$\mathsf{R} = \mathsf{Ind}_{\mathsf{K}}^{\mathsf{G}(M^k)} \mathsf{S}$$

of the Heisenberg group $\mathsf{G}(M^k)$ has weight 1, hence it splits as a sum of copies of the unique irreducible representation $H^0(A, M^k)$ of weight 1:

$$\mathsf{R} = H^0(A, M^k) \oplus \ldots \oplus H^0(A, M^k).$$

We restrict this decomposition to $\mathsf{G}(M)$. By definition, the induced representation R must contain a copy of S as a K-submodule, and therefore also as a $\mathsf{G}(M)$-submodule. We conclude that S must appear in the $\mathsf{G}(M)$-module $H^0(A, M^k)$, as claimed.

Chapter 4

Product abelian surfaces

This chapter appears in the paper [BMOY14] as Section 3.

Relying on the rank one case just established, Theorem 5 is derived by techniques developed in [MO13] and [MO14a]. Specifically, we let

$$X = B \times F \to B$$

be a product of elliptic curves, which we view as an abelian surface elliptically fibered over B. We write f for the class of the fiber over the origin, and σ for the zero section of the fibration. As in [MO14a], stability of sheaves over X is with respect to a polarization

$$H = \sigma + Nf$$

for N large enough. This polarization is suitable in the sense of Friedman [Fri98]. Assuming v and w are vectors with

$$c_1(v) \cdot f = c_1(w) \cdot f = 1,$$

we show that

$$\mathsf{D} : H^0(\mathcal{K}_v, \Theta_w)^{\vee} \to H^0(\mathcal{M}_w, \Theta_v)$$

is an isomorphism.

As in [MO14a], we use a fiberwise Fourier-Mukai transform

$$\mathbf{R}\mathcal{S}^{\dagger} : \mathbf{D}(X) \to \mathbf{D}(X)$$

to move from the rank 1 situation to higher rank Mukai vectors. The kernel of $\mathbf{R}\mathcal{S}^{\dagger}$ is given by the pullback of the normalized Poincaré sheaf

$$\mathcal{P}_F \to F \times F$$

to the product $X \times_B X \cong F \times F \times B$. The Fourier-Mukai transform gives rise to two birational isomorphisms

$$\mathcal{K}_v \dashrightarrow K^{[d_v]}$$

and

$$\mathcal{M}_w \dashrightarrow X^{[d_w]} \times \widehat{X}$$

which are regular in codimension 1. Explicitly, for any $E \in \mathcal{K}_v$ and $F \in \mathcal{M}_w$, away from codimension two loci, Proposition 1 of [MO14a] in conjunction with Theorem 1.1 of [Bri97] shows that

$$\mathbf{R}\mathcal{S}^{\dagger}(E^{\vee}) = I_Z(r\sigma - \chi f)[-1], \tag{4.1}$$

$$\mathbf{R}\mathcal{S}^{\dagger}(F) = I_W^{\vee} \otimes \mathcal{O}(-r'\sigma + \chi' f) \otimes y^{-1}, \tag{4.2}$$

for subschemes

$$Z \in K^{[d_v]}, \ W \in X^{[d_w]}, \ \text{and a line bundle } y \in \widehat{X}.$$

Here, we wrote

$$r = \text{rank } (v), \ \chi = \chi(v), \ r' = \text{rank } (w), \ \chi' = \chi(w).$$

We set

$$L = \mathcal{O}\left((r + r')\sigma - (\chi + \chi')f\right) \implies \chi(L) = d_v + d_w.$$

Now, the key to finishing the proof is the calculation:

$$\mathbb{H}^0(E \otimes^{\mathbf{L}} F) \quad = \quad \mathrm{Hom}_{\mathbf{D}(X)}(E^{\vee}, F) = \mathrm{Hom}_{\mathbf{D}(X)}\left(\mathbf{R}\mathcal{S}^{\dagger}(E^{\vee}), \mathbf{R}\mathcal{S}^{\dagger}(F)\right)$$

$$= \quad \mathrm{Ext}^1(I_Z \otimes y \otimes L, I_W^{\vee}) = \mathrm{Ext}^1(I_W^{\vee}, I_Z \otimes y \otimes L)^{\vee}$$

$$= \quad \mathbb{H}^1(I_W \otimes^{\mathbf{L}} I_Z \otimes y \otimes L)^{\vee}.$$

On the locus (of codimension 2 complement) of non-overlapping (Z, W), the last hyperco-homology group coincides with the regular cohomology group,

$$\mathbb{H}^1(I_W \otimes^{\mathbf{L}} I_Z \otimes y \otimes L) = H^1(I_W \otimes I_Z \otimes y \otimes L).$$

Thus under the birational map

$$\mathsf{K}_v \times \mathsf{M}_w \dashrightarrow K^{[d_v]} \times X^{[d_w]} \times \widehat{X},$$

the two theta divisors

$$\Theta = \{(E, \, F): \ \mathbb{H}^0(E \otimes^{\mathbf{L}} F) \neq 0\} \subset \mathsf{K}_v \times \mathsf{M}_w,$$

and

$$\Theta_L = \{(I_Z, \, I_W, y): \ H^0(I_Z \otimes I_W \otimes y \otimes L) \neq 0\} \subset K^{[d_v]} \times X^{[d_w]} \times \widehat{X}$$

coincide, and the theta line bundles on each factor match up as well. Since in rank 1, Θ_L induces a strange duality isomorphism by Theorem 4, the same must be true about the divisor Θ inducing the map

$$\mathsf{D}: H^0(\mathsf{K}_v, \, \Theta_w)^{\vee} \longrightarrow H^0(\mathsf{M}_w, \, \Theta_v).$$

This completes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \square$

Remark 4.0.1. *The assumption that the rank is at least 3 is made in [MO14a] to justify that equations (4.1) and (4.2) hold in codimension 1. This assumption is however not needed, as we now show. The reader wishing to go on to the proof of generic strange duality contained in the next section may choose to skip this argument.*

To begin, we note that identity (4.2) follows from (4.1) via Grothendieck duality. In turn, equation (4.1) is a consequence of the fact that $\mathbf{R}\mathcal{S}^\dagger(E^\vee)[1]$ is torsion free, cf. Proposition 1 in [MO14a]. We will explain that this assertion holds in codimension 1, in rank 2. To this end, regard the kernel of $\mathbf{R}\mathcal{S}^\dagger$, namely the Poincaré sheaf

$$\mathcal{P} \to X \times_B X,$$

as an object over $X \times X$ via the diagonal embedding

$$X \times_B X \to X \times X.$$

We will prove

Lemma 4.0.2. *For all sheaves E away from a codimension 2 locus in the moduli space, the set*

$$T_E = \{x \in X : \operatorname{Hom}(E, \mathcal{P}_{|X \times \{x\}}) \neq 0\} \subset X$$

is finite.

Assuming the lemma, we show that for all E such that T_E is a finite set, the transform $\mathbf{R}\mathcal{S}^\dagger(E^\vee)[1]$ is a torsion free sheaf. To see this, consider a locally free resolution

$$0 \to V \to W \to E \to 0 \tag{4.3}$$

such that $W = \mathcal{O}_X(-mH)^{\oplus k}$ for sufficiently large m. Then

$$Ext^1(W, \mathcal{P}_{|X \times \{x\}}) = Ext^2(W, \mathcal{P}_{|X \times \{x\}}) = 0,$$

for all $x \in X$. As a consequence, the sheaf

$$\widehat{W} := \mathbf{R}\mathcal{S}^\dagger(W^\vee)$$

is locally free. Next,

$$Ext^2(E, \mathcal{P}_{|X \times \{x\}}) = Hom(\mathcal{P}_{|X \times \{x\}}, E)^\vee = 0,$$

using that E is torsion free and $\mathcal{P}|_{X \times \{x\}}$ is of rank 0. From the exact sequence induced by the resolution (4.3), we conclude that

$$\mathit{Ext}^1(V, \mathcal{P}_{|X \times \{x\}}) = \mathit{Ext}^2(V, \mathcal{P}_{|X \times \{x\}}) = 0$$

for all $x \in X$. Therefore,

$$\widehat{V} := \mathbf{R}\mathcal{S}^\dagger(V^\vee)$$

is locally free as well. The same resolution also shows that we have an exact triangle

$$\mathbf{R}\mathcal{S}^\dagger(E^\vee) \to \widehat{W} \to \widehat{V} \to \mathbf{R}\mathcal{S}^\dagger(E^\vee)[1]$$

which induces an exact sequence in cohomology sheaves

$$0 \to \mathcal{H}^0(\mathbf{R}\mathcal{S}^\dagger(E^\vee)) \to \widehat{W} \xrightarrow{\phi} \widehat{V} \to \mathcal{H}^1(\mathbf{R}\mathcal{S}^\dagger(E^\vee)) \to 0.$$

Note that $\phi_{|\{x\}}$ is injective whenever $x \notin T_E$. Then our assumption implies that ϕ is injective as a morphism of sheaves. Furthermore, $\mathit{Coker}\,\phi$ is torsion free, as claimed. $\qquad\square$

Proof of Lemma 4.0.2. Consider the set

$$\Sigma = \{E : \text{ there exists a fiber } f \text{ such that } E|_f \text{ contains a subbundle of slope } > 1\}.$$

This set has codimension at least 2 in the moduli space by Lemma 5.4 of [BH14]. (A shift by 1 in the slope is necessary to align with the numerical conventions of [BH14].) We will assume that E is chosen outside Σ. Furthermore, we may assume that there is at most one point of the surface where E fails to be locally free. This is always true in the moduli space away from codimension 2.

We claim that in this situation T_E consists of finitely many points. Indeed, let $x \in T_E$. Three cases need to be considered.

(a) First, we rely on the fact that the polarization is suitable. In this case, the restriction of E to a generic fiber is stable. If x lies on such a generic fiber, then as a consequence

of stability, we obtain the vanishing

$$Hom(E, \mathcal{P}_{|X \times \{x\}}) = 0.$$

Therefore in this case $x \notin T_E$.

(b) Assume now that x lies on a fiber f over which the restriction of E is locally free but unstable. In this situation, $E|_f$ splits as

$$E|_f = S_0 \oplus S_1$$

where S_0 is a degree zero line bundle over f, while S_1 has degree 1. Any other splitting type is not allowed by the definition of Σ. Now,

$$Hom(E, \mathcal{P}|_{X \times \{x\}}) = Hom(S_0, \mathcal{P}|_{X \times \{x\}}) \neq 0 \implies S_0 = \mathcal{P}|_{X \times \{x\}}.$$

This shows that x must be the point corresponding to the line bundle S_0. Since by (a), there are only finitely many unstable fibers, we conclude that there are only finitely many choices for x.

(c) Finally, we analyze the case when x lies on a fiber over which E is not locally free. Let s be the unique point where E fails to be locally free, and let f_s be the fiber through s. Then

$$E|_{f_s} = \mathbb{C}_s \oplus F,$$

where F is a rank 2 degree 0 vector bundle over f_s. If F is semistable, there exists an extension

$$0 \to S \to F \to S \to 0$$

where S is a line bundle of degree 0 over f_s. We have

$$Hom(E, \mathcal{P}|_{X \times \{x\}}) = Hom(F, \mathcal{P}|_{X \times \{x\}}) \neq 0 \implies Hom(S, \mathcal{P}|_{X \times \{x\}}) \neq 0$$

$$\implies S = \mathcal{P}|_{X \times \{x\}}.$$

This proves that x is the point of the fiber through s corresponding to S.

To complete the argument, it suffices to show that the situation when F is not semistable corresponds to a codimension 2 subset of the moduli space. To this end, consider the codimension 1 locus $\mathcal{Z}$ of sheaves in the moduli space which fail to be locally free at exactly one point. This is an irreducible subset. Indeed, any sheaf in $\mathcal{Z}$ sits in an exact sequence

$$0 \to E \to E^{\vee\vee} \to \mathbb{C}_s \to 0,$$

with $M = E^{\vee\vee}$ stable locally free of Mukai vector

$$v^{\vee\vee} = v + (0, 0, 1).$$

Letting $\mathcal{M}$ denote the moduli space of such locally free sheaves, there exists a fibration

$$\pi : \mathcal{Z} \to \mathcal{M}$$

whose fibers over M are Quot schemes of length 1 quotients $q : M \to \mathbb{C}_s \to 0$. The sheaf E is recovered uniquely as the kernel of the pair (M, q). Since the fibers of π are irreducible of dimension 3, $\mathcal{Z}$ must be irreducible as well.

Now, for locally free sheaves $M \in \mathcal{M}$, there are finitely many fibers for which $M|_f$ is unstable. Consider

$$\mathcal{Z}^\circ \hookrightarrow \mathcal{Z}$$

the set of pairs $(M, q : M \to \mathbb{C}_s \to 0)$ where s does not lie on an unstable fiber. The restriction of $M|_{f_s}$ is the Atiyah bundle of rank 2 and degree 1. The kernel of q is a torsion free sheaf E which is not locally free at s. In fact, we calculate

$$E|_{f_s} = \mathbb{C}_s \oplus F$$

where F is a subsheaf of degree 0 of the Atiyah bundle $M|_{f_s}$. Since $M|_{f_s}$ is stable, all its proper subbundles have slope ≤ 0. It follows that F is semistable. Thus, to get F's

which are not semistable, we need to select (M, q) from $\mathcal{Z} \setminus \mathcal{Z}^\circ$. Clearly,

$$\mathcal{Z} \setminus \mathcal{Z}^\circ \to \mathcal{M}$$

has projective fibers of dimension 2. Thus, $\mathcal{Z} \setminus \mathcal{Z}^\circ$ has codimension 1 in $\mathcal{Z}$, as claimed. This completes the proof of Lemma 4.0.2 and ends the remark.

$\square$

Chapter 5

Generic strange duality

This chapter appears in the paper [BMOY14] as Section 4.

The isomorphism we established for product abelian surfaces implies strange duality for generic abelian surfaces. This is achieved via degeneration; see also Section 3 of [MO14b]. Specifically, we let $\mathcal{A}$ denote the moduli stack of pairs (X, H) with $H^2 = 2n$, where H is a primitive ample line bundle over X. Consider the universal family

$$\pi : (\mathcal{X}, \mathcal{H}) \to \mathcal{A}.$$

Fix integers χ, χ' and ranks $r, r' \geq 2$. For each $t \in \mathcal{A}$ representing a polarized abelian surface $(\mathcal{X}_t, \mathcal{H}_t)$, consider two orthogonal Mukai vectors

$$v_t = (r, c_1(\mathcal{H}_t), \chi), \ \ w_t = (r', c_1(\mathcal{H}_t), \chi').$$

We form the relative moduli spaces of $\mathcal{H}_t$-semistable sheaves of type v_t and w_t

$$\pi : \mathfrak{K}[v] \to \mathcal{A}, \ \ \pi : \mathfrak{M}[w] \to \mathcal{A}.$$

The product

$$\pi : \mathfrak{K}[v] \times_{\mathcal{A}} \mathfrak{M}[w] \to \mathcal{A}$$

carries the relative Brill-Noether locus

$$\Theta[v, w] = \{(X, H, E, F) : \mathbb{H}^0(X, E \otimes^{\mathbf{L}} F) \neq 0\}$$

obtained as the vanishing of a section of the relative theta line bundle

$$\Theta[w] \boxtimes \Theta[v] \to \mathfrak{K}[v] \times_{\mathcal{A}} \mathfrak{M}[w].$$

Pushing forward to $\mathcal{A}$ via the natural projections π, we obtain the sheaves

$$\mathbb{V} = \pi_\star\left(\Theta[w]\right), \ \ \mathbb{W} = \pi_\star\left(\Theta[v]\right),$$

as well as a section D of $\mathbb{V} \otimes \mathbb{W}$. The constructions are explained in detail in [MO]. Crucial to the specialization procedure which yields generic strange duality is the statement that $\mathbb{V}$ and $\mathbb{W}$ are generically vector bundles of equal rank

$$\frac{d_v^2}{d_v + d_w}\binom{d_v + d_w}{d_v}$$

whose fibers are the spaces of generalized theta functions. This is established in Proposition 2 below. Assuming this result, we let $\mathcal{A}^\circ \hookrightarrow \mathcal{A}$ denote the maximal open locus where the generic rank is achieved. Consider also the Humbert locus

$$\mathcal{S} \hookrightarrow \mathcal{A}$$

of split abelian surfaces

$$(X, H) = (B \times F, L_B \boxtimes L_F),$$

for line bundles $L_B \to B, L_F \to F$ of degrees 1 and n. Just as in Section 3 of [MO14b], Theorem 5 can be rephrased as the statement that

$$\mathcal{S} \hookrightarrow \mathcal{A}^\circ$$

and that furthermore

$$\mathsf{D} : \mathbb{V}^\vee \to \mathbb{W}$$

is an isomorphism along $\mathcal{S}$. To make the above claim, we need to exchange stability relative to a suitable polarization required by Theorem 5 with stability relative to the polarization H (which may lie on a wall). The next section, in particular Proposition 3, shows that the

ensuing moduli spaces agree in codimension 1. We need to pass to the moduli stacks to invoke the proposition, but the corresponding spaces of sections do not change, as explained in Section 3 of [MO14b].

As a consequence, D is an isomorphism generically over $\mathcal{A}^\circ$. Since the generic fibers of $\mathbb{V}$ and $\mathbb{W}$ over $\mathcal{A}^\circ$ are spaces of generalized theta functions, we conclude that generic strange duality holds as in Theorem 6. $\qquad\square$

We now turn to Proposition 2 which was used in the argument above. A general local-freeness statement for the Verlinde sheaves will be proven in Section 6, but in its context, the proposition gives stronger positivity results with a simpler proof. We show

Proposition 2. *Let X be an abelian surface of Picard rank 1, with H the generator of the Néron-Severi group of X. Let*

$$v = (r, H, \chi), \;\; w = (r', d'H, \chi')$$

be two orthogonal vectors of positive rank such that $\chi \neq 0$, $\chi' \leq 0$. Then, for any $F \in \mathcal{K}_w$, the line bundle

$$\Theta_w := \Theta_F \to \mathcal{M}_v$$

is big and nef, hence without higher cohomology. If $\chi' < 0$, then the above line bundle is ample. By restriction, the same results hold for $\Theta_w \to \mathcal{K}_v$.

Proof. In the $K3$ case, reflections along rigid sheaves were used to conclude that $\Theta_w \to \mathcal{M}_v$ is big and nef, hence without higher cohomology, cf. Proposition 4 of [MO14b]. Unlike $K3$ surfaces, abelian surfaces do not admit rigid sheaves. A different argument will be given. The starting point is the following well-known result of Jun Li [Li93]. Specifically, setting

$$w_0 = (0, rH, -2n),$$

the line bundle $\Theta_{w_0} \to \mathcal{M}_v$ is big and nef. We will moreover show that for the vector

$$w_1 = (2n, -\chi H, 0),$$

the line bundle $\Theta_{w_1} \to \mathcal{M}_v$ is also big and nef. Since for $\chi(w) \leq 0$, w is a linear combination with non-negative coefficients of w_0 and w_1, the conclusion follows.

To prove the claim about w_1, we consider two cases depending on the sign of $\chi(v)$. Let us first assume that $\chi(v) < 0$. By Proposition 3.5 of [Yos], the shifted Fourier-Mukai transform Φ with kernel

$$\mathcal{P}[1] \to X \times \widehat{X}$$

induces an isomorphism of moduli spaces

$$\Phi : \mathcal{M}_v \simeq \mathsf{M}_{\widehat{v}} \quad \text{where } \widehat{v} = (-\chi, \widehat{H}, -r) \text{ is a vector on } \widehat{X}.$$

For $\widehat{w} = (0, -\chi\widehat{H}, -2n)$, the bundle

$$\Theta_{\widehat{w}} \to \mathsf{M}_{\widehat{v}}$$

is big and nef, again by Jun Li's result. To conclude, it remains to observe that

$$\Phi^\star \Theta_{\widehat{w}} = \Theta_{w_1},$$

hence the latter line bundle is also big and nef.

When $\chi(v) > 0$, the argument is similar. By Proposition 3.2 of [Yos], we have an isomorphism

$$\Psi : \mathcal{M}_v \simeq \mathsf{M}_{\widehat{v}}, \quad \widehat{v} = (\chi, \widehat{H}, r)$$

induced by the composition of the Fourier-Mukai transform with kernel $\mathcal{P}$ with the dualization. Under this isomorphism, Jun Li's bundle $\Theta_{\widehat{w}}$, where $\widehat{w} = (0, \chi\widehat{H}, -2n)$, corresponds to Θ_{w_1}. $\qquad\square$

5.1 Variation of polarization for the moduli space of Gieseker sheaves

This chapter appears in the paper [BMOY14] as Section 5.

Let X be an arbitrary abelian surface, and fix a Mukai vector

$$v := (r, \xi, a) \in H^*(X, \mathbb{Z})$$

with $r > 0$. For an ample divisor H on X, denote by

$$\mathfrak{M}(v), \ \mathfrak{M}_H(v)^{ss} \text{ and } \mathfrak{M}_H(v)^{\mu-ss}$$

the stacks of all sheaves, of Gieseker H-semistable sheaves, and of slope H-semistable sheaves respectively – all of type v.

We are concerned with moduli spaces of sheaves when Gieseker stability varies: we show that they agree in codimension 1 each time a wall is crossed. This fact was used in the degeneration argument of Section 5 to exchange the suitable polarization with the polarization determined by the first Chern class.

First, for generic polarizations, the dimension of the moduli space is given by the following Lemma 4.3.2 in [MYY11]:

Lemma 5.1.1. *If H is general with respect to v, that is, H does not lie on a wall with respect to v, then*

$$\dim \mathfrak{M}_H(v)^{ss} = \begin{cases} \langle v, v \rangle + 1, & \langle v, v \rangle > 0 \\ \langle v, v \rangle + \ell, & \langle v, v \rangle = 0 \end{cases}, \tag{5.1}$$

where $\ell = \gcd(r, \xi, a)$.

For the purposes of Chapter 5, we also need to analyze the situation when the polarization may lie on a wall. To this end, let H_1 be an ample divisor on X which belongs to a wall W with respect to v and H an ample divisor which belongs to an adjacent chamber. Then Gieseker H-semistable sheaves are slope H_1-semistable

$$\mathfrak{M}_H(v)^{ss} \hookrightarrow \mathfrak{M}_{H_1}^{ss}(v) \hookrightarrow \mathfrak{M}_{H_1}(v)^{\mu-ss}.$$

All these stacks have dimension $\langle v, v \rangle + 1$ by Lemma 3.8 of [KY08]. We estimate the codimension of

$$\mathfrak{M}_{H_1}(v)^{ss} \setminus \mathfrak{M}_H(v)^{ss}.$$

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Specifically, we prove

Proposition 3. *Assume that v is a Mukai vector of positive rank with the property that there are no isotropic vectors u of positive rank such that $\langle v, u \rangle = 1$ or 2. Then,*

$$(\langle v, v \rangle + 1) - \dim(\mathfrak{M}_{H_1}(v)^{ss} \setminus \mathfrak{M}_H(v)^{ss}) \geq 2. \tag{5.2}$$

Therefore, in this situation, $\mathfrak{M}_H(v)^{ss}$ is independent of the choice of ample line bundle H (generic or on a wall) away from codimension 2.

The same statement holds true for the moduli stack $\mathfrak{K}_H(v)^{ss}$ of sheaves with fixed determinant and fixed determinant of the Fourier-Mukai.

Proof. The proof is essentially contained in Proposition 4.3.4 of [MYY11], but since specific aspects of the argument are used below, we give an outline for the benefit of the reader. Let E be a Gieseker H_1-semistable sheaf, which is however not Gieseker H-semistable. In particular E is slope H_1-semistable. Consider the Harder-Narasimhan filtration relative to H

$$0 \subset F_1 \subset F_2 \subset \cdots \subset F_s = E.$$

By definition, the reduced H-Hilbert polynomials of F_i/F_{i-1} are strictly decreasing. In particular, the H-slopes are decreasing as well. In turn, this implies

$$\mu_{H_1}(F_1) \geq \mu_{H_1}(F_2/F_1) \geq \ldots \geq \mu_{H_1}(F_s/F_{s-1}),$$

and therefore

$$\mu_{H_1}(F_1) \geq \mu_{H_1}(F_2) \geq \ldots \geq \mu_{H_1}(F_s) = \mu_{H_1}(E).$$

Since E is slope H_1-semistable, we must have equality throughout

$$\mu_{H_1}(F_1) = \mu_{H_1}(F_2) = \ldots = \mu_{H_1}(E).$$

Equivalently, writing

$$v(F_i/F_{i-1}) = v_i \text{ so that } v = \sum_{i=1}^{s} v_i,$$

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we obtain

$$\frac{c_1(v_i) \cdot H_1}{\mathrm{rk}\ v_i} = \frac{c_1(v) \cdot H_1}{\mathrm{rk}\ v}, \quad 1 \le i \le s. \tag{5.3}$$

Let $\mathcal{F}_H(v_1, v_2, \ldots, v_s)$ be the stack of the Harder-Narashimhan filtrations

$$0 \subset F_1 \subset F_2 \subset \cdots \subset F_s = E, \ \ E \in \mathfrak{M}(v) \tag{5.4}$$

such that the quotients F_i/F_{i-1}, $1 \le i \le s$ are semistable with respect to H and

$$v(F_i/F_{i-1}) = v_i. \tag{5.5}$$

Thus

$$\mathfrak{M}_{H_1}(v)^{\mu\text{-}ss} \setminus \mathfrak{M}_H(v)^{ss} = \cup_{v_1,\ldots,v_s} \mathcal{F}_H(v_1, v_2, \ldots, v_s),$$

where (5.3) is satisfied. Then Lemma 5.3 in [KY08] implies

$$\dim \mathcal{F}_H(v_1, v_2, \ldots, v_s) = \sum_{i=1}^{s} \dim \mathfrak{M}_H(v_i)^{ss} + \sum_{i<j} \langle v_i, v_j \rangle. \tag{5.6}$$

Write $v_i = \ell_i v_i'$ where v_i' is a primitive Mukai vector. It is shown in Proposition 4.3.4 of [MYY11] that for all i, j we have

$$\langle v_i', v_j' \rangle \ge 3$$

unless either v_i' or v_j' is isotropic, and in this case $\langle v_i', v_j' \rangle \ge 1$. We estimate

$$
\begin{aligned}
(\langle v, v \rangle + 1) \ \ - \ \ & \dim(\mathfrak{M}_{H_1}(v)^{\mu\text{-}ss} \setminus \mathfrak{M}_H(v)^{ss}) \\
= \ \ & (\langle v, v \rangle + 1) - \sum_{i<j} \langle v_i, v_j \rangle - \sum_{i=1}^{s} \dim \mathfrak{M}_H(v_i)^{ss} \\
= \ \ & \sum_{i>j} \langle v_i, v_j \rangle - \sum_{i=1}^{s} (\dim \mathfrak{M}_H(v_i)^{ss} - \langle v_i, v_i \rangle) + 1 \\
\ge \ \ & \sum_{i>j} \ell_i \ell_j \langle v_i', v_j' \rangle - \sum_{i=1}^{s} \ell_i + 1 \ge \sum_{i>j} \ell_i \ell_j - \sum_{i} \ell_i + 1 \ge 2.
\end{aligned}
$$

Indeed, the above inequality is satisfied for $s \ge 4$. The cases $s = 2$ and $s = 3$ need to be considered separately. The detailed analysis is contained in Proposition 4.3.4 of [MYY11]. The only possible exceptions correspond to

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- $s = 2$, $\ell_1 = 1$, $\ell_2 = \ell$, v_2' isotropic, $\langle v_1', v_2' \rangle = 1$;

- $s = 2$, $\ell_1 = 1$, $\ell_2 = 1$, v_1' isotropic, $\langle v_1', v_2' \rangle = 2$;

- $s = 3$, $\ell_1 = \ell_2 = \ell_3 = 1$, $v = v_1' + v_2' + v_3'$, v_i' isotropic, $\langle v_i', v_j' \rangle = 1$.

In all cases, taking $u = v_1'$, we obtain $\langle v, u \rangle = 1$ or 2, which contradicts our assumption. For the final claim about the moduli space $\mathfrak{K}_H(v)^{ss}$, we repeat the proof above. The only modification is the dimension estimate (5.6) which follows by going over the argument in [KY08]. $\qquad\square$

Lemma 5.1.2. *Assume that*

$$\langle v, v \rangle > 4 \ rank \ (v).$$

Then no isotropic vector u of positive rank satisfying $\langle v, u \rangle = 1$ or 2 occurs as Mukai vector of a quotient in a Harder-Narasimhan filtration of a sheaf of type v. Therefore, the moduli spaces

$$\mathfrak{M}_H(v)^{ss} \ and \ \mathfrak{K}_H(v)^{ss}$$

are independent of the polarization H in codimension 1.

Proof. Assume that there exists an isotropic vector u as above such that $\langle v, u \rangle = 1$ or 2. In this situation, we have

$$\frac{c_1(u) \cdot H_1}{\mathrm{rk} \ u} = \frac{c_1(v) \cdot H_1}{\mathrm{rk} \ v} \implies \left(\frac{c_1(u)}{\mathrm{rk} \ u} - \frac{c_1(v)}{\mathrm{rk} v} \right) \cdot H_1 = 0.$$

Using the Hodge index theorem, we conclude that

$$\left(\frac{c_1(u)}{\mathrm{rk} \ u} - \frac{c_1(v)}{\mathrm{rk} v} \right)^2 \leq 0.$$

By direct calculation, or via Lemma 1.1 of [KY08], we obtain

$$
\begin{aligned}
\langle v, u \rangle &= -\frac{\mathrm{rk}(v) \cdot \mathrm{rk}(u)}{2} \left(\frac{c_1(u)}{\mathrm{rk} \ u} - \frac{c_1(v)}{\mathrm{rk} v} \right)^2 + \frac{\mathrm{rk} \ (u)}{\mathrm{rk} \ (v)} \cdot \frac{\langle v, v \rangle}{2} + \frac{\mathrm{rk} \ (v)}{\mathrm{rk} \ (u)} \cdot \frac{\langle u, u \rangle}{2} \\
&\geq \frac{\mathrm{rk} \ (u)}{\mathrm{rk} \ (v)} \cdot \frac{\langle v, v \rangle}{2} > 2 \ \mathrm{rk} \ (u) \geq 2.
\end{aligned}
$$

This contradiction completes the proof. $\qquad\square$

Remark 5.1.3. *The Lemma above applies to the particular situation of a product abelian surface $X = B \times F$ considered in Section 5. We assume here that B, F are not isogenous, so that the section σ and the fiber class f generate the Néron-Severi group. Then, for Mukai vectors*

$$v = (r, \sigma + nf, \chi), \ w = (r', \sigma + nf, \chi')$$

with $\chi, \chi' < 0$ we obtain

$$\langle v, v \rangle = 2n - 2r\chi > -2r\chi \geq 4r,$$

as required in order to apply the Lemma.

The only exception may be the case $\chi = -1$ which will be treated separately. In this situation, we claim that there are no walls between the polarizations

$$H = \sigma + nf, \quad H' = \sigma + Nf,$$

where N is taken sufficiently large to ensure that H' is suitable. Indeed, assuming otherwise, consider a wall defined by an isotropic Mukai vector u such that

$$\langle v, u \rangle = 1 \ or \ 2.$$

In fact, possibly doubling u, it suffices to analyze the case $\langle v, u \rangle = 2$. Let

$$H_0 = \sigma + kf, \ k \geq n,$$

be an ample divisor on this wall, where $k \in \mathbb{Q}$. By definition, the vector u appears as the Mukai vector of a quotient in the Harder-Narasimhan filtration for H_0. Setting $u = (p, \eta, q)$ with $p > 0$, we obtain from (5.3) that

$$\left(\frac{\eta}{p} - \frac{H}{r} \right) \cdot H_0 = 0 \implies (r\eta - pH) \cdot H_0 = 0.$$

Writing

$$r\eta - pH = a\sigma + bf,$$

we calculate

$$(r\eta - pH) \cdot H_0 = (a\sigma + bf) \cdot (\sigma + kf) = 0 \implies b = -ak.$$

Consequently,

$$(r\eta - pH)^2 = (a\sigma + bf)^2 = 2ab = -2a^2k \le -2k, \qquad (5.7)$$

unless $a = b = 0$. This particular situation can be analyzed by exactly the same methods; we leave the verification to the reader. In any case, the conditions that u is isotropic and $\langle v, u \rangle = 2$ translate into

$$\eta^2 = 2pq, \quad \eta \cdot H = -p + qr + 2,$$

respectively. With this understood, we compute the left hand side of (5.7)

$$(r\eta - pH)^2 = r^2\eta^2 + p^2H^2 - 2pr(\eta \cdot H) = 2np^2 + 2pr(p - 2) \ge 0 > -2k$$

with the only possible exception $p = 1$. In this case, the above calculation yields

$$(r\eta - pH)^2 = 2n - 2r.$$

By orthogonality,

$$2n = -r'\chi - r\chi' \ge r + r' > r$$

which implies

$$(r\eta - pH)^2 = 2n - 2r > -2n \ge -2k.$$

This contradicts (5.7), showing that there is no wall separating H from a suitable polarization.

$\square$

Chapter 6

The Verlinde sheaves are locally free

This chapter appears in the paper [BMOY14] as Section 6.

The goal of this section is to prove Theorem 7. We show that for any (X, H), the dimension of the space of sections of the theta line bundles is given by the expected formula (2.2) for a very general class of Mukai vectors. This holds even without knowing the vanishing of higher cohomology. As a consequence, the Verlinde sheaves $\mathbb{V}$ and $\mathbb{W}$ used in the degeneration argument of Section 5 are in fact locally free over the entire moduli space $\mathcal{A}$ of pairs (X, H). The result should be compared to Proposition 2 of Section 5. The *generic* local-freeness yielded by Proposition 2 was sufficient for proving our main Theorem 6. By contrast, Theorem 4 gives *global* local-freeness in great generality, and will be useful for future strange duality studies.

We split the theorem into two statements with proofs of different flavors. First, we show

Proposition 4. *Let (X, H) be a polarized abelian surface. Assume that*

$$v = (r, dH, \chi), \ w = (r', d'H, \chi')$$

are orthogonal primitive Mukai vectors of ranks $r, r' \geq 2$ such that

(i) $d, d' > 0$;

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(ii) $\chi < 0, \ \chi' < 0$.

Assume furthermore that if $(d, \chi) = (1, -1)$, then (X, H) is not a product of two elliptic curves. We have

$$h^0(\mathcal{K}_v, \Theta_w) = \chi(\mathcal{K}_v, \Theta_w) = \frac{d_v^2}{d_v + d_w}\binom{d_v + d_w}{d_v}.$$

In the same context, the Proposition implies the requisite statement for the moduli space $\mathcal{M}_v$:

Proposition 5. *In the setup of Proposition 4, for any representative $F \in \mathcal{K}_w$ we have*

$$h^0(\mathcal{M}_v, \Theta_F) = \chi(\mathcal{M}_v, \Theta_F) = \frac{d_w^2}{d_v + d_w}\binom{d_v + d_w}{d_v}.$$

6.1 Proof of Proposition 4.

We begin by explaining the strategy of the proof when $\mathcal{K}_v$ is smooth. The key point is Lemma 6.1.1 below which shows that $\Theta_w \to \mathcal{K}_v$ is movable, hence (big and) nef on a smooth birational model of $\mathcal{K}_v$, cf. Theorem 7 of [HT09]. The birational models of $\mathcal{K}_v$ arise as moduli spaces of Bridgeland stable objects. The dimension calculation is carried out on the moduli space of Bridgeland stable objects, where the higher cohomology vanishes. The Proposition follows since wall-crossings do not change the dimension of the space of sections. The case when $\mathcal{K}_v$ may be singular requires first to desingularize the moduli space. The above argument can then be repeated on a symplectic resolution.

Let us elaborate the discussion. As already remarked, the proof uses moduli spaces of Bridgeland stable objects. Specifically, we consider stability conditions $\sigma = \sigma_{s,t} = (Z_{s,t}, \mathcal{A}_{s,t})$, for $t > 0$, corresponding to central charges

$$Z_{s,t}(E) = \langle \exp((s + it)H), v(E) \rangle.$$

The heart $\mathcal{A}_{s,t}$ has as objects certain 2-step complexes, and is obtained as a tilt of the abelian category of coherent sheaves on X at a certain torsion pair; the exact definition will not be

used below, but we refer the reader to [Bri08] for details. We form the moduli spaces $\mathcal{M}_v(\sigma)$ of σ-semistable objects of type v. The moduli space comes equipped with the Albanese map

$$\mathsf{a} : \mathcal{M}_v(\sigma) \to X \times \widehat{X},$$

and we write $\mathcal{K}_v(\sigma)$ for the Albanese fiber.

We begin by analyzing the case $\mathcal{K}_v$ smooth. The following observations (a)-(c) are useful for the argument.

(a) In the large volume limit $t >> 0$, Bridgeland stability with respect to $\sigma_{s,\infty} := \sigma_{s,t}$ coincides with Gieseker stability, cf. [Bri08], Section 14.

The next remarks (b)-(c) are contained in the recent papers [MYY13] and [Yos12b]. For $K3$ surfaces, the similar statements are found in [BM14a].

(b) The space of stability conditions admits a wall and chamber decomposition, so that the moduli spaces are constant in each chamber, but they undergo explicit birational transformations as walls are crossed. These birational transformations are regular in codimension 1.

For the next remark, observe that the theta map (2.1) gives an isomorphism

$$\Theta : (v^\vee)^\perp \to \mathrm{Pic}(\mathcal{K}_v(\sigma)),$$

in such a fashion that the Beauville-Bogomolov form on the right hand side corresponds to the Mukai pairing on the left hand side. Two basic (real) cones of divisors are necessary for our purposes. First, the positive cone

$$\mathrm{Pos}(\mathcal{K}_v(\sigma)) \hookrightarrow \mathrm{Pic}(\mathcal{K}_v(\sigma))_\mathbb{R}$$

can be expressed via the Beauville-Bogomolov form

$$\mathrm{Pos}(\mathcal{K}_v(\sigma)) = \{x : \langle x, x \rangle > 0, \ \langle x, A \rangle > 0 \text{ for a fixed ample divisor } A \text{ over } \mathcal{K}_v(\sigma)\}.$$

Second, the movable cone

$$\mathrm{Mov}(\mathcal{K}_v(\sigma)) \hookrightarrow \mathrm{Pic}(\mathcal{K}_v(\sigma))_{\mathbb{R}}$$

is generated by divisors whose stable base locus has codimension 2 or higher. Positive movable divisors are big and nef on some smooth birational models, cf. Theorem 7 of [HT09]. In our context, we have the following result obtained via the study of the movable cone in [MYY11]:

(c) A positive movable divisor

$$\mathrm{Mov}(\mathcal{K}_v(\sigma_{s,\infty})) \cap \mathrm{Pos}(\mathcal{K}_v(\sigma_{s,\infty}))$$

is identified, under the birational wall crossings of (b), with a big and nef divisor on a smooth moduli space $\mathcal{K}_v(\sigma_{s,t})$ of Bridgeland stable objects:

$$\{\Theta_w \to \mathcal{K}_v(\sigma_{s,\infty})\} \quad \longleftrightarrow \quad \{\Theta_w \to \mathcal{K}_v(\sigma_{s,t})\}.$$

(Note that the Mukai vector w labeling the theta line bundle may undergo Weyl reflections when crossing divisorial walls in $(v^{\vee})^{\perp}$. However, since there are no divisorial walls within the movable chamber, w does not change in the present setting.)

The essential ingredient is then provided by the following

Lemma 6.1.1. *For v and w as in Proposition 4, the line bundle $\Theta_w \to \mathcal{K}_v(\sigma_{s,\infty})$ belongs to the positive movable cone.*

As a consequence of remarks (a)-(c) and of the lemma, we note

$$h^0(\mathcal{K}_v(\sigma_{s,\infty}), \Theta_w) = h^0(\mathcal{K}_v(\sigma_{s,t}), \Theta_w) = \chi(\mathcal{K}_v(\sigma_{s,t}), \Theta_w).$$

By the same argument as for the usual Gieseker stability, as in Proposition 1 of [MO08a], we further have

$$\chi(\mathcal{K}_v(\sigma_{s,t}), \Theta_w) = \frac{d_v^2}{d_v + d_w}\binom{d_v + d_w}{d_v}.$$

We conclude that

$$h^0(\mathcal{K}_v(\sigma_{s,\infty}), \Theta_w) = \frac{d_v^2}{d_v + d_w}\binom{d_v + d_w}{d_v},$$

as claimed in Proposition 4. □

Proof of Lemma 6.1.1. We begin by noting that Θ_w is positive in the Gieseker chamber. Indeed,

$$\langle \Theta_w, \Theta_w \rangle = \langle w, w \rangle > 0.$$

For the second inequality, an ample divisor on the moduli space $\mathcal{K}_v(\sigma_{s,\infty})$ is constructed in [LP96]; see also Remark 8.1.12 of [HL97]. This divisor takes the form Θ_a for

$$a = (r, rmH, -2mnd - \chi), \quad \text{where } m >> 0.$$

Recalling that $w = (r', d'H, \chi')$, we have

$$\langle \Theta_w, \Theta_a \rangle = \langle w, a \rangle = 2nm(d'r + dr') - r\chi' + r'\chi > 0,$$

as needed.

We will now show that for the vector

$$w_1 = (2nd, -\chi H, 0)$$

the line bundle Θ_{w_1} belongs to the closure of the movable cone for the Gieseker chamber. We will combine this with a well-known result of Jun Li [Li93]. For the vector

$$w_0 = (0, rH, -2nd)$$

the associated theta line bundle

$$\Theta_{w_0} \to \mathcal{K}_v(\sigma_{s,\infty})$$

is big and nef, so in particular it is in the closure of the movable cone. Notice now that the vector w is a positive linear combination of w_0 and w_1,

$$w = \frac{1}{2nd}(-\chi' w_0 + r' w_1),$$

hence Θ_w is movable.

To prove the claim about w_1, we will use the description of the movable cone given in [MYY13] and [BM14a]. Specifically, we consider the hyperplanes in $\overline{\mathrm{Pos}}(\mathcal{K}_v(\sigma_{s,\infty}))$ given by

$$\Theta((u^\vee)^\perp \cap (v^\vee)^\perp), \ \ 1 \leq \langle v, u \rangle \leq 2, \ \langle u, u \rangle = 0.$$

The movable cone is cut out by these hyperplanes. To prove that Θ_{w_1} and Θ_{w_0} belong to the same chamber, it suffices to show that

$$\langle w_0, u^\vee \rangle \geq 0 \iff \langle w_1, u^\vee \rangle \geq 0,$$

whenever u is isotropic and $1 \leq \langle v, u \rangle \leq 2$. The first inequality above will in fact turn out strict for rank 3 or higher.

We assume $r > 2$ first. Let us write $u = (p, \eta, q)$ where

$$\eta^2 = 2pq, \ \ p, q \in \mathbb{Z}.$$

Changing u into $-u$, we may furthermore assume that $p \geq 0$ and $\langle v, u \rangle = \pm 1, \pm 2$. Recalling that $v = (r, dH, \chi)$, we calculate

$$\langle v, u \rangle = d(H \cdot \eta) - p\chi - qr = \pm 1 \text{ or } \pm 2. \tag{6.1}$$

We compute

$$\langle w_0, u^\vee \rangle \geq 0 \iff -r(H \cdot \eta) + 2ndp \geq 0. \tag{6.2}$$

Similarly,

$$\langle w_1, u^\vee \rangle \geq 0 \iff \chi(H \cdot \eta) - 2ndq \geq 0. \tag{6.3}$$

We therefore need to show that

$$-r(H \cdot \eta) + 2ndp \geq 0 \iff \chi(H \cdot \eta) - 2ndq \geq 0.$$

We consider first the case when $p = 0$. Then, replacing u by $-u$ we may assume that $H \cdot \eta \geq 0$. In fact, $H \cdot \eta = 0$ is impossible by (6.1) since $r > 2$. Therefore, $H \cdot \eta > 0$. In this

situation, (6.2) is false. We argue that (6.3) is false as well. Assuming otherwise, we have

$$\chi(H \cdot \eta) \geq 2ndq \implies q < 0.$$

This is however incompatible with (6.1) which reads

$$d(H \cdot \eta) + r(-q) = \pm 1, \pm 2,$$

which is impossible for $r > 2$.

The crux of the argument is the case $p > 0$. In this situation, we distinguish the following subcases:

(i) Assume $H \cdot \eta = 0$. By the Hodge index theorem $\eta^2 \leq 0$ hence

$$pq = \frac{\eta^2}{2} \leq 0 \implies q \leq 0.$$

This shows that both (6.2) and (6.3) are true at the same time.

(ii) Assume $H \cdot \eta < 0$. In this case, (6.2) is true. We prove that (6.3) is true as well. Assuming otherwise, we obtain that

$$\chi(H \cdot \eta) - 2ndq < 0.$$

In particular $q > 0$ and multiplying by $p > 0$ we see that

$$\frac{p\chi}{2nd}(H \cdot \eta) < pq = \frac{\eta^2}{2}.$$

By the Hodge index theorem, we have

$$\eta^2 \leq \frac{(H \cdot \eta)^2}{2n}.$$

The above inequality becomes

$$\frac{p\chi}{2nd}(H \cdot \eta) < \frac{(H \cdot \eta)^2}{4n} \implies (H \cdot \eta) < \frac{2p\chi}{d}.$$

We obtain therefore

$$d(H \cdot \eta) - p\chi - qr < 2p\chi - p\chi - qr = p\chi - qr < -2,$$

using $\chi < 0$ and $q > 0$. This contradicts (6.1). Thus (6.3) must be true as well.

(iii) Assume $H \cdot \eta > 0$. Equation (6.1) implies that $q \geq 0$. In this case, the inequality (6.3) is false. We argue that (6.2) is false as well. Assume otherwise, so that

$$r(H \cdot \eta) \leq 2ndp \implies \frac{rq}{2nd}(H \cdot \eta) \leq pq = \frac{\eta^2}{2}.$$

Again by the Hodge index theorem, we have

$$\eta^2 \leq \frac{(H \cdot \eta)^2}{2n}$$

yielding

$$\frac{rq}{2nd}(H \cdot \eta) \leq pq = \frac{\eta^2}{2} \leq \frac{(H \cdot \eta)^2}{4n} \implies 2rq \leq d(H \cdot \eta).$$

We obtain

$$d(H \cdot \eta) - p\chi - rq \geq 2rq - p\chi - rq = rq - p\chi > 2$$

if $q > 0$, contradicting (6.1). When $q = 0$, equation (6.1) yields

$$d(H \cdot \eta) - p\chi = \pm 1, \pm 2,$$

which implies $d(H \cdot \eta) = 1, p\chi = -1$. Therefore $(d, \chi) = (1, -1)$ and $H \cdot \eta = 1, \eta^2 = 0$. In this case, (X, H) is a product of elliptic curves, which is not allowed.[1]

When $r = 2$, the same argument goes through with the only exception corresponding to the case

$$p = 0, \ H \cdot \eta = 0.$$

Since $\eta^2 = 2pq = 0$ we obtain $\eta = 0$ by the Hodge index theorem. This yields the isotropic vector $u = (0, 0, 1)$. In fact, w_0 lies on the wall determined by u, hence we cannot pin down

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[1]To see that X is a product, write $\tau = H - n \cdot \eta$. Therefore,

$$\eta^2 = \tau^2 = 0, \ \eta \cdot \tau = 1.$$

In this situation, η and τ are represented by two elliptic curves E and F, cf. Proposition 2.3 in [Kan94]. The sum morphism

$$s : E \times F \to X$$

must be an isogeny. The preimage of the origin corresponds to the intersection $E \cap F$, hence s must be an isomorphism.

on which side of the wall w_1 lies. To remedy this problem, we replace w_0 by the vector

$$a = (r, rmH, -2mnd - \chi) = mw_0 + (r, 0, -\chi)$$

which we have already seen to give an ample theta bundle for $m \gg 0$. For the vector $u = (0, 0, 1)$, direct computation shows

$$\langle a, u \rangle < 0, \ \langle w_1, u \rangle < 0,$$

hence w_1 and a are also on the same side of the wall determined by u.

This completes the analysis, and therefore the proof when $\mathcal{K}_v$ is smooth.

However, $\mathcal{K}_v$ may be singular when the polarization H is not generic. In this situation, for any $\beta \in \mathrm{NS}(X)_\mathbb{Q}$, we consider the moduli space of β-twisted H-semistable sheaves. Recall that a sheaf E is β-twisted H-semistable provided that

(i) for all subsheaves $F \subset E$, we have

$$\frac{c_1(F) \cdot H}{\mathrm{rk}(F)} \leq \frac{c_1(E) \cdot H}{\mathrm{rk}(E)};$$

(ii) if equality holds in (i), then

$$\frac{\chi(F) - c_1(F) \cdot \beta}{\mathrm{rk}(F)} \leq \frac{\chi(E) - c_1(E) \cdot \beta}{\mathrm{rk}(E)}.$$

We form the moduli space $\mathcal{K}_\beta(v)$ of β-twisted H-semistable sheaves. In fact, remark (a) above applies here as well, and consequently, $\mathsf{K}_\beta(v)$ can be viewed as a moduli space of Bridgeland's stable objects. In addition, if β is appropriately chosen, then the moduli space $\mathcal{K}_\beta(v)$ consists of stable sheaves only, and therefore is a smooth non-empty holomorphic symplectic manifold; see for instance Lemma 5.4 of [Abe00]. Furthermore, Lemma 5.5 in [Abe00] shows that there is a surjective morphism

$$\pi : \mathcal{K}_\beta(v) \to \mathcal{K}_v,$$

which is therefore a symplectic resolution. As a consequence of Proposition 1.3 of [Bea00] we have

$$\mathbf{R}\pi_\star \mathcal{O}_{\mathsf{K}_\beta(v)} = \mathcal{O}_{\mathcal{K}_v}.$$

Now, as the moduli space $\mathsf{K}_\beta(v)$ consists of stable sheaves only, it carries a theta line bundle Θ_w. Furthermore, the line bundle Θ_w descends to the singular moduli space $\mathcal{K}_v$, which may contain strictly semistables. This is a consequence of Kempf's lemma and is shown to hold true in Theorem 8.1.5 of [HL97]. The essential point is that $c_1(v) = dH$ is a multiple of the polarization. As a corollary,

$$H^0(\mathcal{K}_v, \Theta_w) = H^0(\mathsf{K}_\beta(v), \pi^\star\Theta_w) = H^0(\mathsf{K}_\beta(v), \Theta_w).$$

We claim that Θ_w is movable over the smooth moduli space $\mathcal{K}_\beta(v)$. In fact, the argument we presented in the untwisted case carries over to the twisted situation. An essential ingredient of the proof is that Jun Li's line bundle is big and nef. This continues to hold over $\mathcal{K}_\beta(v)$ by pullback, at least for β chosen as above. Alternatively, ample divisors are constructed in Lemma 5.5.2 of [MYY13]. Since Θ_w is movable, we conclude that

$$h^0(\mathsf{K}_\beta(v), \Theta_w) = \chi(\mathsf{K}_\beta(v), \Theta_w) = \frac{d_v^2}{d_v + d_w}\binom{d_v + d_w}{d_v},$$

as claimed. This completes the proof.

$\square$

Remark 6.1.2. *The argument above also remains valid in ranks 0 and 1. Consequently, the dimension calculation of Proposition 4 holds true for all primitive orthogonal Mukai vectors*

$$v = (r, dH, \chi), \ w = (r', d'H, \chi') \ \text{with } r, r' \geq 0, \ d, d' > 0, \ \chi, \chi' < 0,$$

with the extra assumption that

- *(X, H) is not a product when $(d, \chi) = (1, -1)$ or when $(r, d) = (1, 1)$.*

6.2 Proof of Proposition 5.

To prove the Proposition, we use the diagram

$$
\begin{array}{ccc}
\mathcal{K}_v \times X \times \widehat{X} & \xrightarrow{\Phi_v} & \mathcal{M}_v \\
\downarrow{\scriptstyle p} & & \downarrow{\scriptstyle a} \\
X \times \widehat{X} & \xrightarrow{\Psi_v} & X \times \widehat{X}
\end{array} \quad .
$$

Here, $\Phi_v : \mathcal{K}_v \times X \times \widehat{X} \to \mathcal{M}_v$ is defined as

$$
\Phi_v(E, x, y) = t_x^\star E \otimes y,
$$

and

$$
a : \mathcal{M}_v \to X \times \widehat{X}
$$

is the Albanese map. Both Φ_v and Ψ_v are étale of degree d_v^4 [Yos], [MO08a]. In fact, it is proved in [Yos] that

$$
\Psi_v(x, y) = (-\chi x - d\varphi_{\widehat{H}}(y),\ d\varphi_H(x) + ry),
$$

where as usual

$$
\widehat{H} \to \widehat{X}
$$

is the inverse determinant of the Fourier-Mukai transform of H, and φ_H, $\varphi_{\widehat{H}}$ denote the Mumford homomorphisms. This explicit expression will however not be needed below.

Fix $F \in \mathcal{K}_w$. We have

$$
\Phi_v^\star \Theta_F = \Theta_w \boxtimes \mathcal{L}
$$

for a line bundle $\mathcal{L} \to X \times \widehat{X}$. It is shown in Proposition 4 of [MO08a] that

$$
\chi(\mathcal{L}) = d_v^2 d_w^2.
$$

In fact, by Lemma 1 in [Opr11], up to numerical equivalence we have

$$
\mathcal{L} = H^a \boxtimes \widehat{H}^b \otimes \mathcal{P}^c, \tag{6.4}
$$

where $\mathcal{P} \to X \times \widehat{X}$ is the Poincaré bundle, and

$$a = -(\chi d' + \chi' d), \ \ b = rd' + r'd, \ \ c = dd'n + r'\chi = -dd'n - r\chi'.$$

In consequence of the assumptions $\chi, \chi' < 0$ and $d, d' > 0$, and also of the calculation

$$abn - c^2 = d_v d_w > 0,$$

we obtain the inequalities

$$a > 0, \ \ b > 0, \ \ abn > c^2.$$

These inequalities ensure that the line bundle $\mathcal{L}$ is ample. To see this, we use the special form of the Nakai-Moishezon criterion for ampleness in the context of abelian varieties, as stated on page 77 of [BL04]. Specifically, for abelian varieties, the criterion asserts that it is enough to check ampleness numerically on hyperplanes and intersections of hyperplanes under any fixed projective embedding, such as the one induced by $H + \widehat{H}$. A direct calculation then shows that a line bundle $\mathcal{L} \to X \times \widehat{X}$ of the form (6.4) is ample if and only if the three inequalities above are satisfied. In consequence, $\mathcal{L}$ has no higher cohomology.

With this understood, we write with the aid of Proposition 4

$$h^0(\mathcal{K}_v \times X \times \widehat{X}, \Theta_w \boxtimes \mathcal{L}) = h^0(\mathcal{K}_v, \Theta_w)h^0(X \times \widehat{X}, \mathcal{L}) = \chi(\mathcal{K}_v, \Theta_w)\chi(X \times \widehat{X}, \mathcal{L}) \quad (6.5)$$

$$= \frac{d_v^2}{d_v + d_w}\binom{d_v + d_w}{d_v} \cdot (d_v d_w)^2.$$

On the other hand,

$$h^0(\mathcal{K}_v \times X \times \widehat{X}, \Phi_v^\star \Theta_F) = h^0(\mathcal{M}_v, (\Phi_v)_\star \Phi_v^\star \Theta_F) = \sum_\tau h^0(\mathcal{M}_v, \Theta_F \otimes \mathsf{a}^\star \mathbb{L}_\tau) \quad (6.6)$$

where

$$(\Psi_v)_\star \mathcal{O} = \bigoplus_\tau \mathbb{L}_\tau,$$

over $X \times \widehat{X}$. The line bundles $\mathbb{L}_\tau$ appearing in the decomposition above are indexed by the characters $\tau \in \widehat{\mathsf{G}}_v$ of the group

$$\mathsf{G}_v = \mathrm{Ker} \ \Psi_v.$$

We claim that

Lemma 6.2.1. *For each character τ of G_v, there exists an automorphism $f_\tau : \mathcal{M}_v \to \mathcal{M}_v$ such that*

$$\Theta_F \otimes \mathsf{a}^\star \mathbb{L}_\tau = f_\tau^\star \Theta_F.$$

By the lemma, we therefore have

$$h^0(\mathcal{M}_v, \Theta_F \otimes \mathsf{a}^\star \mathbb{L}_\tau) = h^0(\mathcal{M}_v, \Theta_F)$$

hence by (6.6) we obtain

$$h^0(\mathcal{K}_v \times X \times \widehat{X}, \Phi_v^\star \Theta_F) = \deg \Psi_v \cdot h^0(\mathcal{M}_v, \Theta_F) = d_v^4 \cdot h^0(\mathcal{M}_v, \Theta_F).$$

This implies via (6.5) that

$$h^0(\mathcal{M}_v, \Theta_F) = \frac{d_w^2}{d_v + d_w}\binom{d_v + d_w}{d_v},$$

establishing Proposition 5. $\qquad\square$

Proof of Lemma 6.2.1. We consider the group

$$K(\mathcal{L}) \hookrightarrow X \times \widehat{X}$$

of pairs (x, y) leaving $\mathcal{L}$ invariant by translation

$$t_{(x,y)}^\star \mathcal{L} \simeq \mathcal{L}.$$

The group $K(\mathcal{L})$ has $\chi(\mathcal{L})^2 = (d_v d_w)^4$ elements.

For each pair $(x, y) \in K(\mathcal{L})$, we define the automorphism

$$f_{(x,y)} : \mathcal{M}_v \to \mathcal{M}_v$$

given by

$$f_{(x,y)}(E) = t_x^\star E \otimes y.$$

We show that for $(x, y) \in K(\mathcal{L})$ we can find a line bundle $\mathbb{L}_\tau \in \widehat{\mathsf{G}}_v$ such that

$$f_{(x,y)}^\star \Theta_F = \Theta_F \otimes \mathsf{a}^\star \mathbb{L}_\tau. \tag{6.7}$$

Indeed, the two lines bundles $f_{(x,y)}^\star \Theta_F$ and Θ_F both restrict to Θ_w on each fiber of the Albanese map a, hence for some line bundle $\mathbb{L} \to X \times \widehat{X}$ we have

$$f_{(x,y)}^\star \Theta_F = \Theta_F \otimes \mathsf{a}^\star \mathbb{L}.$$

It remains to explain that

$$\Psi_v^\star \mathbb{L} = \mathcal{O},$$

or equivalently that

$$\Phi_v^\star f_{(x,y)}^\star \Theta_F = \Phi_v^\star \Theta_F.$$

Direct calculation shows that over $\mathcal{K}_v \times X \times \widehat{X}$ we have

$$f_{(x,y)} \circ \Phi_v = \Phi_v \circ \left(1, t_{(x,y)}\right).$$

Therefore

$$
\begin{aligned}
\Phi_v^\star f_{(x,y)}^\star \Theta_F &= \left(1, t_{(x,y)}\right)^\star \Phi_v^\star \Theta_F = \left(1, t_{(x,y)}\right)^\star (\Theta_w \boxtimes \mathcal{L}) \\
&= \Theta_w \boxtimes t_{(x,y)}^\star \mathcal{L} = \Theta_w \boxtimes \mathcal{L} = \Phi_v^\star \Theta_F.
\end{aligned}
$$

As a consequence of (6.7), there exists a group homomorphism

$$\alpha : K(\mathcal{L}) \to \widehat{\mathsf{G}}_v.$$

To complete the proof of the Lemma, we argue that α is surjective. Since

$$\text{order } K(\mathcal{L}) = (d_v d_w)^4, \ \text{order} \mathsf{G}_v = d_v^4$$

it suffices to prove that

$$\text{order } \mathrm{Ker}\, \alpha = d_w^4.$$

In fact, we claim that

$$\mathrm{Ker}\, \alpha \simeq \mathsf{G}_w, \tag{6.8}$$

where G_w is the kernel of the morphism Ψ_w in the diagram

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$$\begin{array}{ccc} \mathcal{K}_w \times X \times \widehat{X} & \xrightarrow{\Phi_w} & \mathcal{M}_w \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle a} \\ X \times \widehat{X} & \xrightarrow{\Psi_w} & X \times \widehat{X} \end{array} \quad .$$

Here, $\Phi_w : \mathcal{K}_w \times X \times \widehat{X} \to \mathcal{M}_w$ is defined as

$$\Phi_w(G, x, y) = t^\star_{-x} G \otimes y,$$

and

$$a : \mathcal{M}_w \to X \times \widehat{X}$$

is the Albanese map

$$a(G) = (\det \widehat{G} \otimes \widehat{H}^{d'}, \det G \otimes H^{-d'}).$$

Furthermore, just as above, Φ_w and Ψ_w both have degree d_w^4. To prove (6.8), note that

$$(x, y) \in \operatorname{Ker} \alpha \iff f^\star_{(x, y)} \Theta_F = \Theta_F \iff \Theta_{t^\star_{-x} F \otimes y} = \Theta_F.$$

By [MO08a], the last equality happens if and only if

$$\det(t^\star_{-x} F \otimes y) = \det F \text{ and } \det (\widehat{t^\star_{-x} F \otimes} y) = \det \widehat{F}$$

$$\iff (a \circ \Phi_w)(F, x, y) = 0 \iff \Psi_w(x, y) = 0 \iff (x, y) \in \mathsf{G}_w,$$

as claimed. The proof of the lemma is completed. $\qquad\square$

Part II

Part II: Nef cones of Hilbert schemes of points on surfaces

The content of Part II is the content of the paper [BHL$^+$15].

Chapter 7

Motivation and setup

If X is a projective variety, the cone $\mathrm{Amp}(X) \subset N^1(X)$ of ample divisors controls the various projective embeddings of X. It is one of the most important invariants of X, and carries detailed information about the geometry of X. Its closure is the *nef cone* $\mathrm{Nef}(X)$, which is dual to the Mori cone of curves (see for example [Laz04]). In this paper, we will study the nef cone of the Hilbert scheme of points $X^{[n]}$, where X is a smooth projective surface over $\mathbb{C}$.

Nef divisors on Hilbert schemes of points on surfaces $X^{[n]}$ are sometimes easy to construct by classical methods. If L is an $(n-1)$-very ample line bundle on X, then for any $Z \in X^{[n]}$ we have an inclusion $H^0(L \otimes I_Z) \to H^0(L)$ which defines a morphism from $X^{[n]}$ to the Grassmannian $G(h^0(L) - n, h^0(L))$. The pullback of an ample divisor on the Grassmannian is nef on $X^{[n]}$. It is frequently possible to construct extremal nef divisors by this method. For example, this method completely computes the nef cone of $X^{[n]}$ when X is a del Pezzo surface of degree ≥ 2 or a Hirzebruch surface (see [ABCH12], [BC13]). Unfortunately, this approach to computing the nef cone is insufficient in general. At the very least, to study nef cones of more interesting surfaces it would be necessary to study an analog of k-very ampleness for higher rank vector bundles, which is considerably more challenging than line bundles.

More recently, many nef cones have been computed by making use of Bridgeland stability conditions and the Positivity Lemma of Bayer and Macrì (see [Bri07], [Bri08], [AB13], and [BM14b] for background on these topics, which will be reviewed in Section 8). Let $\mathbf{v} = \mathrm{ch}(I_Z) \in K_0(X)$, where $Z \in X^{[n]}$. In the stability manifold $\mathrm{Stab}(X)$ for X there is an open *Gieseker chamber* $\mathcal{C}$ such that if $\sigma \in \mathcal{C}$ then $M_\sigma(\mathbf{v}) \cong X^{[n]}$, where $M_\sigma(\mathbf{v})$ is the moduli space of σ-semistable objects with invariants $\mathbf{v}$. The Positivity Lemma associates to any $\sigma \in \overline{\mathcal{C}}$ a nef divisor on $X^{[n]}$. Stability conditions in the boundary $\partial\mathcal{C}$ frequently give rise to extremal nef divisors. The Positivity Lemma also classifies the curves orthogonal to a nef divisor constructed in this way, and so gives a tool for checking extremality.

The stability manifold is rather large in general, so computation of the full Gieseker chamber can be unwieldy. We deal with this problem by focusing on a small *slice* of the stability manifold parameterized by a half-plane. Up to scale, the corresponding divisors in $N^1(X)$ form an affine ray. The nef cone $\mathrm{Nef}(X^{[n]})$ is spanned by a codimension 1 subcone identified with $\mathrm{Nef}(X)$ and other more interesting classes which are positive on curves contracted by the Hilbert–Chow morphism. Since $\mathrm{Nef}(X^{[n]})$ is convex, we can study $\mathrm{Nef}(X^{[n]})$ by looking at positivity properties of divisors along rays in $N^1(X)$ starting from a class in $\mathrm{Amp}(X) \subset \mathrm{Nef}(X^{[n]})$. The Positivity Lemma gives us an effective criterion for testing when divisors along the ray are nef.

The slices of the stability manifold that we consider are given by a pair of divisors (H, D) on X with H ample and $-D$ effective. The following is a weak version of one of our main theorems.

Theorem 8. *Let X be a smooth projective surface. If $n \gg 0$, then there is an extremal nef divisor on $X^{[n]}$ coming from the (H, D)-slice. It can be explicitly computed if both the intersection pairing on $\mathrm{Pic}(X)$ and the set of effective classes in $\mathrm{Pic}(X)$ are known. An orthogonal curve class is given by n points moving in a g_n^1 on a curve of a particular class.*

See Section 9 for more explicit statements, especially Corollary 3 and Theorem 15. Stronger

statements can also be shown under strong assumptions on $\mathrm{Pic}(X)$; for example, we study the Picard rank one case in detail in Section 10. Recall that if X is surface of irregularity $q := H^1(\mathcal{O}_X) = 0$ then $N^1(X^{[n]})$ is spanned by the divisor B of nonreduced schemes and divisors $L^{[n]}$ induced by divisors $L \in \mathrm{Pic}(X)$; see Section 8.1 for details.

Theorem 9. *Let X be a smooth projective surface with $\mathrm{Pic}X \cong \mathbb{Z}H$, where H is an ample divisor. Let $a > 0$ be the smallest integer such that aH is effective. If*

$$n \geq \max\{a^2 H^2, p_a(aH) + 1\},$$

then $\mathrm{Nef}(X^{[n]})$ is spanned by the divisor $H^{[n]}$ and the divisor

$$\frac{1}{2} K_X^{[n]} + \left(\frac{a}{2} + \frac{n}{aH^2}\right) H^{[n]} - \frac{1}{2} B. \qquad (*)$$

An orthogonal curve class is given by letting n points move in a g_n^1 on a curve in X of class aH.

Note that in the Picard rank 1 case the divisor class $(*)$ is frequently of the form $\lambda H^{[n]} - \frac{1}{2}B$ for a non-integer number $\lambda \in \mathbb{Q}$. Any divisor constructed from an $(n-1)$-very ample line bundle will be of the form $\lambda H^{[n]} - \frac{1}{2}B$ with $\lambda \in \mathbb{Z}$, so in general the edge of the nef cone cannot be obtained from line bundles in this way.

The required lower bound on n in Theorem 9 can be improved in specific examples where special linear series on hyperplane sections are better understood.

Theorem 10. *Let X be one of the following surfaces:*

1. *a very general hypersurface in $\mathbb{P}^3$ of degree $d \geq 4$, or*

2. *a very general degree d cyclic branched cover of $\mathbb{P}^2$ of general type.*

In either case, $\mathrm{Pic}(X) \cong \mathbb{Z}H$ with H effective. Suppose $n \geq d-1$ in the first case, and $n \geq d$ in the second case. Then $\mathrm{Nef}(X^{[n]})$ is spanned by $H^{[n]}$ and the divisor class $(*)$ with $a = 1$.

Finally, in Section 11 we compute the nef cone of $X^{[n]}$ where X is a smooth del Pezzo surface of degree 1 and $n \geq 2$ is arbitrary. This computation was an open problem posed by Bertram and Coskun in [BC13]; they noted that the method of k-very ample line bundles would not be sufficient to prove the expected answer. Since X has Picard rank 9, this computation makes full use of the general methods developed in Section 9. If $C \subset X$ is a reduced, irreducible curve which admits a g_n^1, we write $C_{[n]}$ for the curve in the Hilbert scheme $X^{[n]}$ given by letting n points move in a g_n^1 on C.

Theorem 11. *Let X be a smooth del Pezzo surface of degree 1. The Mori cone of curves* $\mathrm{NE}(X^{[n]})$ *is spanned by the 240 classes $E_{[n]}$ given by (-1)-curves $E \subset X$, the class of a curve contracted by the Hilbert–Chow morphism, and the class $F_{[n]}$, where $F \in |-K_X|$ is an anticanonical curve. The nef cone is determined by duality.*

Many previous authors have used Bridgeland stability conditions to study nef cones and wall-crossing for Hilbert schemes $X^{[n]}$ and moduli spaces of sheaves $M_H(\mathbf{v})$ for various classes of surfaces. For instance, the program was studied for $\mathbb{P}^2$ in [ABCH12], [CH14a], [BMW14], and [LZ13], for Hirzebruch and del Pezzo surfaces in [BC13], abelian surfaces in [YY14] and [MM13], K3 surfaces in [BM14b], [?] and [HT10], and Enriques surfaces in [Nue14b]. Our results unify several of these approaches. Additionally, nef cones were classically studied in the context of k-very ample line bundles in papers such as [?], [BS91], [BFS89], and [CG90].

Chapter 8

Preliminaries

Throughout the paper, we let X be a smooth projective surface over $\mathbb{C}$.

8.1 Divisors and curves on $X^{[n]}$

For simplicity we assume that X has irregularity $q = h^1(\mathcal{O}_X) = 0$ in this subsection. By work of Fogarty [Fog68], the Hilbert scheme $X^{[n]}$ is a smooth projective variety of dimension $2n$ which resolves the singularities in the symmetric product $X^{(n)}$ via the Hilbert–Chow morphism $X^{[n]} \to X^{(n)}$. A line bundle L on X induces the S_n-equivariant line bundle $L^{\boxtimes n}$ on X^n which descends to a line bundle $L^{(n)}$ on the symmetric product $X^{(n)}$. The pullback of $L^{(n)}$ by the Hilbert–Chow morphism $X^{[n]} \to X^{(n)}$ defines a line bundle on $X^{[n]}$ which we will denote by $L^{[n]}$. Intuitively, if $L \cong \mathcal{O}_X(D)$ for a reduced effective divisor $D \subset X$, then $L^{[n]}$ can be represented by the divisor $D^{[n]}$ of schemes $Z \subset X$ which meet D.

Fogarty shows that

$$\mathrm{Pic}(X^{[n]}) \cong \mathrm{Pic}(X) \oplus \mathbb{Z}(B/2),$$

where $\mathrm{Pic}(X) \subset \mathrm{Pic}(X^{[n]})$ is embedded by $L \mapsto L^{[n]}$ and B is the locus of non-reduced schemes, i.e., the exceptional divisor of the Hilbert–Chow morphism [Fog73]. Tensoring by the real numbers, the Neron–Severi space $N^1(X^{[n]})$ is therefore spanned by $N^1(X)$ and B.

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There are also curve classes in $X^{[n]}$ induced by curves in X. Two different constructions are immediate. Let $C \subset X$ be a reduced and irreducible curve.

1. There is a curve $\tilde{C}_{[n]}$ in $X^{[n]}$ given by fixing $n-1$ general points of X and letting an nth point move along C.

2. If C admits a g_n^1, i.e., a degree n map to $\mathbb{P}^1$, then the fibers of $C \to \mathbb{P}^1$ give a rational curve $\mathbb{P}^1 \to X^{[n]}$. We write $C_{[n]}$ for this class.

These constructions preserve intersection numbers, in the sense that if $D \subset X$ is a divisor and $C \subset X$ is a curve then

$$D^{[n]} \cdot \tilde{C}_{[n]} = D^{[n]} \cdot C_{[n]} = D \cdot C.$$

Part of the nef cone $\mathrm{Nef}(X^{[n]})$ is easily described in terms of the nef cone of X. If D is an ample divisor, then $D^{(n)}$ is ample so $D^{[n]}$ is nef. In the limit, we find that if D is nef then $D^{[n]}$ is nef. Conversely, if D is not nef then there is an irreducible curve C with $D \cdot C < 0$, so $D^{[n]} \cdot \tilde{C}_{[n]} < 0$ and $D^{[n]}$ is not nef. Under the Fogarty isomorphism,

$$\mathrm{Nef}(X^{[n]}) \cap N^1(X) = \mathrm{Nef}(X).$$

The hyperplane $N^1(X) \subset N^1(X^{[n]})$ is orthogonal to any curve contracted by the Hilbert–Chow morphism, so all the divisors in $\mathrm{Nef}(X) \subset \mathrm{Nef}(X^{[n]})$ are extremal. Since B is the exceptional locus of the Hilbert–Chow morphism, we see that any nef class must have non-positive coefficient of B. After scaling, then, we see that computation of the cone $\mathrm{Nef}(X^{[n]})$ reduces to describing the nef classes of the form $L^{[n]} - \frac{1}{2}B$ lying outside $\mathrm{Nef}(X) \subset \mathrm{Nef}(X^{[n]})$.

8.2 Bridgeland stability conditions

We now recall some basic definitions and properties of Bridgeland stability conditions. We fix a polarization $H \in \mathrm{Pic}(X)_{\mathbb{R}}$. For any divisor $D \in \mathrm{Pic}(X)_{\mathbb{R}}$ the twisted Chern character

$\mathrm{ch}^D = e^{-D}\mathrm{ch}$ can be expanded as

$$\mathrm{ch}_0^D = \mathrm{ch}_0,$$

$$\mathrm{ch}_1^D = \mathrm{ch}_1 - D\mathrm{ch}_0,$$

$$\mathrm{ch}_2^D = \mathrm{ch}_2 - D \cdot \mathrm{ch}_1 + \frac{D^2}{2}\mathrm{ch}_0.$$

Recall that a Bridgeland stability condition is a pair $\sigma = (Z, \mathcal{A})$ where $Z : K_0(X) \to \mathbb{C}$ is an additive homomorphism and $\mathcal{A} \subset D^b(X)$ is the heart of a bounded t-structure. In particular, $\mathcal{A}$ is an abelian category. Moreover, Z maps any non trivial object in $\mathcal{A}$ to the upper half plane or the negative real line. The σ-slope function is defined by

$$\nu_\sigma = -\frac{\Re Z}{\Im Z},$$

and σ-(semi)stability of objects of $\mathcal{A}$ is defined in terms of this slope function. More technical requirements are the existence of Harder–Narasimhan filtrations and the support property. We recommend Bridgeland's article [Bri07] for a more precise definition. The support property is well explained in Appendix A of [BMS14].

In the case of surfaces, Bridgeland [Bri08] and Arcara–Bertram [AB13] showed how to construct Bridgeland stability conditions in a slice corresponding to a choice of an ample divisor $H \in \mathrm{Pic}(X)_\mathbb{R}$ and arbitrary twisting divisor $D \in \mathrm{Pic}(X)_\mathbb{R}$. The classical Mumford slope function for twisted Chern characters is defined by

$$\mu_{H,D} = \frac{H \cdot \mathrm{ch}_1^D}{H^2\mathrm{ch}_0^D},$$

where torsion sheaves are interpreted as having positive infinite slope. Given a real number $\beta \in \mathbb{R}$ there are two categories defined as

$$\mathcal{T}_\beta = \{E \in \mathrm{Coh}(X) : \text{any quotient } E \twoheadrightarrow G \text{ satisfies } \mu_{H,D}(G) > \beta\},$$

$$\mathcal{F}_\beta = \{E \in \mathrm{Coh}(X) : \text{any subsheaf } F \hookrightarrow E \text{ satisfies } \mu_{H,D}(F) \leq \beta\}.$$

A new heart of a bounded t-structure is defined as the extension closure $\mathcal{A}_\beta := \langle \mathcal{F}_\beta[1], \mathcal{T}_\beta \rangle$. We fix an additional positive real number α and define the homomorphism as

$$Z_{\beta,\alpha} = -\text{ch}_2^{D+\beta H} + \frac{\alpha^2 H^2}{2}\text{ch}_0^{D+\beta H} + iH \cdot \text{ch}_1^{D+\beta H}.$$

The pair $\sigma_{\beta,\alpha} := (Z_{\beta,\alpha}, \mathcal{A}_\beta)$ is then a Bridgeland stability condition. The (H, D)-*slice* of stability conditions is the family of stability conditions $\{\sigma_{\beta,\alpha} : \beta, \alpha \in \mathbb{R}, \alpha > 0\}$ parameterized by the (β, α) upper half plane.

Definition 8.2.1. Fix a set of invariants $\mathbf{v} \in K_0(X)$.

1. Let $\mathbf{w} \in K_0(X)$ be a vector such that $\mathbf{v}$ and $\mathbf{w}$ do not have the same $\sigma_{\beta,\alpha}$-slope everywhere in the (H, D)-slice. The *numerical wall* for $\mathbf{v}$ given by $\mathbf{w}$ is the set of points (β, α) where $\mathbf{v}$ and $\mathbf{w}$ have the same $\sigma_{\beta,\alpha}$-slope.

2. A numerical wall for $\mathbf{v}$ given by a vector $\mathbf{w}$ as above is a *wall* (or *actual wall*) if there is a point (β, α) on the wall and an exact sequence $0 \to F \to E \to G \to 0$ in $\mathcal{A}_\beta$, where $\text{ch}F = \mathbf{w}$, $\text{ch}E = \mathbf{v}$, and F, E, G are $\sigma_{\beta,\alpha}$-semistable objects (of the same $\sigma_{\beta,\alpha}$-slope).

We write $K_{\text{num}}(X)$ for the numerical Grothendieck group of classes in $K_0(X)$ modulo numerical equivalence. Note that numerical walls for $\mathbf{v} \in K_0(X)$ only depend on the numerical class of $\mathbf{v}$, while actual walls a priori depend on $c_1(\mathbf{v}) \in \text{Pic}(X)$. The structure of walls in a slice is heavily restricted by Bertram's Nested Wall Theorem. This was first observed for Picard rank one with $D = 0$, but the proof immediately generalizes by replacing ch by ch^D everywhere.

Theorem 12 ([Mac14])**.** *Let* $\mathbf{v} \in K_0(X)$.

1. *Numerical walls for* $\mathbf{v}$ *can either be semicircles with center on the* β-*axis or the unique vertical line given by* $\beta = \mu_{H,D}(\mathbf{v})$. *Moreover, the apex of each semicircle lies on the hyperbola* $\Re Z_{\beta,\alpha}(\mathbf{v}) = 0$.

2. *Numerical walls for* **v** *are disjoint, and the semicircular walls on either side of the vertical wall are nested.*

3. *If W_1 and W_2 are two semicircular numerical walls left of the vertical wall with centers $(s_{W_1}, 0)$ and $(s_{W_2}, 0)$, then W_2 is nested inside W_1 if and only if $s_{W_1} < s_{W_2}$.*

4. *Suppose $0 \to F \to E \to G \to 0$ is an exact sequence destabilizing an object E with $\mathrm{ch}(E) = \mathbf{v}$ at a point (β, α) on a numerical wall W, in the sense that all three objects have the same $\sigma_{\beta,\alpha}$-slope and this is an exact sequence in $\mathcal{A}_\beta$. Then it is an exact sequence of objects in $\mathcal{A}_{\beta'}$ with the same $\sigma_{\beta',\alpha'}$-slope for all $(\beta', \alpha') \in W$. That is, E is destabilized along the entire wall.*

8.3   Slope and discriminant

The explicit geometry of walls is frequently best understood in terms of slopes and discriminants; the formulas presented here previously appeared in [CH14b] in the context of $\mathbb{P}^2$. When the rank is nonzero, we define

$$\Delta_{H,D} = \frac{1}{2}\mu_{H,D}^2 - \frac{\mathrm{ch}_2^D}{H^2\mathrm{ch}_0^D}.$$

The Bogomolov inequality gives $\Delta_{H,D}(E) \geq 0$ whenever E is an (H, D)-twisted Giesker semistable sheaf. Observe that $\Delta_{H,D+\beta H} = \Delta_{H,D}$ for every $\beta \in \mathbb{R}$. A straightforward calculation shows that for vectors of nonzero rank the slope function for the stability condition $\sigma_{\beta,\alpha}$ in the (H, D)-slice is given by

$$\nu_{\sigma_{\beta,\alpha}} = \frac{(\mu_{H,D} - \beta)^2 - \alpha^2 - 2\Delta_{H,D}}{(\mu_{H,D} - \beta)} \tag{8.1}$$

Suppose $\mathbf{v}, \mathbf{w}$ are two classes with positive rank, and let their slopes and discriminants be $\mu_{H,D}, \Delta_{H,D}$ and $\mu'_{H,D}, \Delta'_{H,D}$, respectively. The numerical wall W in the (H, D)-slice where $\mathbf{v}$ and $\mathbf{w}$ have the same slope is computed as follows.

- If $\mu_{H,D} = \mu'_{H,D}$ and $\Delta_{H,D} = \Delta'_{H,D}$, then $\mathbf{v}$ and $\mathbf{w}$ have the same slope everywhere in the slice, so there is no numerical wall.

- If $\mu_{H,D} = \mu'_{H,D}$ and $\Delta_{H,D} \neq \Delta'_{H,D}$, then W is the vertical wall $\beta = \mu_{H,D}$.

- If $\mu_{H,D} \neq \mu'_{H,D}$, then Equation (8.1) implies W is the semicircle with center $(s_W, 0)$ and radius ρ_W, where

$$s_W = \frac{1}{2}(\mu_{H,D} + \mu'_{H,D}) - \frac{\Delta_{H,D} - \Delta'_{H,D}}{\mu_{H,D} - \mu'_{H,D}}, \tag{8.2}$$

$$\rho_W^2 = (s_W - \mu_{H,D})^2 - 2\Delta_{H,D} \tag{8.3}$$

provided that the expression defining ρ_W^2 is positive; if it is negative then the wall is empty.

Notice that if $\Delta_{H,B}(\mathbf{v}) \geq 0$ then numerical walls for $\mathbf{v}$ left of the vertical wall accumulate at the point

$$\left(\mu_{H,D}(\mathbf{v}) - \sqrt{2\Delta_{H,D}(\mathbf{v})}, 0\right) \tag{8.4}$$

as their radii go to 0.

8.4 Nef divisors and the Positivity Lemma

In this section, we describe the Positivity Lemma of Bayer and Macrì. Let $\sigma = (Z, \mathcal{A})$ be a stability condition on X, $\mathbf{v} \in K_{\mathrm{num}}(X)$ and S a proper algebraic space of finite type over $\mathbb{C}$. Let $\mathcal{E} \in D^b(X \times S)$ be a flat family of σ-semistable objects of class $\mathbf{v}$, i.e., for every $\mathbb{C}$-point $p \in S$, the derived restriction $\mathcal{E}|_{\pi_S^{-1}(\{p\})}$ is σ-semistable of class $\mathbf{v}$. Then Bayer and Macrì define a numerical divisor class $D_{\sigma, \mathcal{E}} \in N^1(S)$ on the space S by assigning its intersection with any projective integral curve $C \subset S$:

$$D_{\sigma, \mathcal{E}} \cdot C = \Im\left(-\frac{Z((p_X)_* \mathcal{E}|_{C \times X})}{Z(\mathbf{v})}\right).$$

The Positivity Lemma shows that this divisor inherits positivity properties from the homomorphism Z, and classifies the curve classes orthogonal to the divisor. Recall that two σ-semistable objects are *S-equivalent* with respect to σ if their sets of Jordan–Hölder factors are the same.

Theorem 13 (Positivity Lemma, [BM14b, Lemma 3.3]). *The divisor $D_{\sigma,\mathcal{E}} \in N^1(S)$ is nef. Moreover, if $C \subset S$ is a projective integral curve then $D_{\sigma,\mathcal{E}} \cdot C = 0$ if and only if two general objects parameterized by C are S-equivalent with respect to σ.*

Our primary use of the Positivity Lemma is to attempt to construct extremal nef divisors on Hilbert schemes of points. Thus it is important to recover Hilbert schemes of points as Bridgeland moduli spaces. Recall that a torsion-free coherent sheaf E is (H, D)-*twisted Gieseker semistable* if for every $F \subset E$ we have

$$\frac{\chi(F \otimes \mathcal{O}_X(mH - D))}{\mathrm{rk}(F)} \leq \frac{\chi(E \otimes \mathcal{O}_X(mH - D))}{\mathrm{rk}(E)}$$

for all $m \gg 0$, where the Euler characteristic is computed formally via Riemann–Roch; see [MW97]. For any class $\mathbf{v} \in K_0(X)$, there are projective moduli spaces $M_{H,D}(\mathbf{v})$ of S-equivalence classes of (H, D)-twisted Gieseker semistable sheaves with class $\mathbf{v}$. If $\mathbf{v} = (1, 0, -n)$ is the Chern character of an ideal sheaf of n points then $M_{H,D}(\mathbf{v}) = X^{[n]}$. Note that if the irregularity of X is nonzero, then it is crucial to fix the determinant.

Fix an (H, D)-slice in the stability manifold, and fix a vector $\mathbf{v} \in K_0(X)$ with positive rank. If β lies to the left of the vertical wall $\beta = \mu_{H,D}(\mathbf{v})$ for $\mathbf{v}$, then for $\alpha \gg 0$ the moduli space coincides with a twisted Gieseker moduli space.

Proposition 6 (The large volume limit [Bri08, Mac14]). *Fix divisors (H, D) giving a slice in $\mathrm{Stab}(X)$. Let $\mathbf{v} \in K_0(X)$ be a vector with positive rank, and let $\beta \in \mathbb{R}$ be such that $\mu_{H,D}(\mathbf{v}) > \beta$. If $E \in \mathcal{A}_\beta$ has $\mathrm{ch}(E) = \mathbf{v}$ then E is $\sigma_{\beta,\alpha}$-semistable for all $\alpha \gg 0$ if and only if E is an $(H, D - \frac{1}{2}K_X)$-twisted Gieseker semistable sheaf.*

Moreover, in the quadrant of the (H, D)-slice left of the vertical wall there is a largest semicircular wall for $\mathbf{v}$, called the Gieseker wall. For all (β, α) between this wall and the vertical wall, the moduli space $M_{\sigma_{\beta,\alpha}}(\mathbf{v})$ coincides with the moduli space $M_{H, D - K_X/2}(\mathbf{v})$ of $(H, D - \frac{1}{2}K_X)$-twisted Gieseker semistable sheaves.

We use these results as follows. Let $\mathbf{v} = (1, 0, -n) \in K_0(X)$ be the vector for the Hilbert scheme $X^{[n]}$, and let σ_+ be a stability condition in the (H, D)-slice lying above the Gieseker wall, so that $M_{\sigma_+}(\mathbf{v}) \cong X^{[n]}$. Let $\mathcal{E}/(X \times X^{[n]})$ be the universal ideal sheaf, and let σ_0 be a stability condition on the Gieseker wall. By the definition of the Gieseker wall, $\mathcal{E}$ is a family of σ_0-semistable objects, so there is an induced nef divisor $D_{\sigma_0, \mathcal{E}}$ on $X^{[n]}$. Furthermore, curves orthogonal to $D_{\sigma_0, \mathcal{E}}$ are understood in terms of destabilizing sequences along the wall, so it is possible to test for extremality.

Chapter 9

Gieseker walls and the nef cone

Fix an ample divisor $H \in \mathrm{Pic}(X)$ with $H^2 = d$ and an *antieffective* divisor D. In this section we study the nef divisor arising from the Gieseker wall (i.e., the largest wall where some ideal sheaf is destabilized) in the slice of the stability manifold given by the pair (H, D). We first compute the Gieseker wall, and then investigate when the corresponding nef divisor is in fact extremal.

9.1 Bounding higher rank walls

The main difficulty in computing extremal rays of the nef cone is to show that a destabilizing subobject along the Gieseker wall is a line bundle, and not some higher rank sheaf. We first prove a lemma which generalizes [CH14a, Proposition 8.3] from $X = \mathbb{P}^2$ to an arbitrary surface. We prove the result in slightly more generality than we will need here as we expect it to be useful in future work.

Lemma 9.1.1. *Let σ_0 be a stability condition in the (H, D)-slice, and suppose*

$$0 \to F \to E \to G \to 0$$

is an exact sequence of σ_0-semistable objects of the same σ_0-slope, where E is an (H, D)-

twisted Gieseker semistable torsion-free sheaf. If the map $F \to E$ of sheaves is not injective, then the radius ρ_W of the wall W defined by this sequence satisfies

$$\rho_W^2 \leq \frac{(\min\{\mathrm{rk}(F) - 1, \mathrm{rk}(E)\})^2}{2\mathrm{rk}(F)} \Delta_{H,D}(E).$$

Proof. The proof is similar to the proof in [CH14a] given in the case of $\mathbb{P}^2$; we present it for completeness. The object F is a torsion-free sheaf by the standard cohomology sequence and the fact that the heart of the t-structure in the slice we are working in consists of objects which only have nonzero cohomology sheaves in degrees 0 and -1. The exact sequence along W gives an exact sequence of sheaves

$$0 \to K \to F \to E \to C \to 0$$

of ranks k, f, e, c, respectively. By assumption, $k, f, e > 0$. Let $(s_W, 0)$ be the center of W. As F is in the categories $\mathcal{T}_\beta$ whenever (β, α) is on W, we find $\mu_{H,D}(F) \geq s_W + \rho_W$, so

$$df(s_W + \rho_W) \leq df\mu_{H,D}(F) = \mathrm{ch}_1^D(F) \cdot H = (\mathrm{ch}_1^D(K) + \mathrm{ch}_1^D(E) - \mathrm{ch}_1^D(C)) \cdot H$$

$$= dk\mu_{H,D}(K) + de\mu_{H,D}(E) - \mathrm{ch}_1^D(C) \cdot H.$$

Similarly, $K \in \mathcal{F}_\beta$ along W, so $\mu_{H,D}(K) \leq s_W - \rho_W$ and

$$df(s_W + \rho_W) \leq dk(s_W - \rho_W) + de\mu_{H,D}(E) - \mathrm{ch}_1^D(C) \cdot H,$$

which gives

$$d(k + f)\rho_W \leq d(k - f)s_W + de\mu_{H,D}(E) - \mathrm{ch}_1^D(C) \cdot H. \tag{9.1}$$

We now wish to eliminate the term $\mathrm{ch}_1^D(C) \cdot H$ in Inequality (9.1). If C is either 0 or torsion, then $\mathrm{ch}_1^D(C) \cdot H \geq 0$ and $-e = k - f$, and we deduce

$$(k + f)\rho_W \leq (k - f)(s_W - \mu_{H,D}(E)). \tag{9.2}$$

Suppose instead that C is not torsion. Since C is a quotient of the semistable sheaf E, we have $\mu_{H,D}(C) \geq \mu_{H,D}(E)$, so $\mathrm{ch}_1^D(C) \cdot H = dc\mu_{H,D}(C) \geq dc\mu_{H,D}(E)$. As $k - f = c - e$, we find that Inequality (9.2) also holds in this case.

Both sides of Inequality (9.2) are positive, so squaring both sides gives

$$(k+f)^2\rho_W^2 \leq (k-f)^2(s_W - \mu_{H,D}(E))^2.$$

The formula (8.3) for ρ_W^2 shows this is equivalent to

$$(k+f)^2\rho_W^2 \leq (k-f)^2\left(\rho_W^2 + 2\Delta_{H,D}(E)\right),$$

from which we obtain

$$\rho_W^2 \leq \frac{(k-f)^2}{2kf}\Delta_{H,D}(E).$$

Since $k = f - e + c$, we see that $k \geq \max\{1, f - e\}$. By taking derivatives in k, we see that $\frac{(k-f)^2}{2kf}$ is decreasing for $k + f > 0$, and so the maximum possible value of the right-hand side must occur when $k = \max\{1, f - e\}$. The denominator will be at least $2f$ in this case, and the numerator is $\min\{(f-1)^2, e^2\}$. The result follows. □

For our present work we will only need the next consequence of Lemma 9.1.1 which follows immediately from computing $\Delta_{H,D}(I_Z)$.

Corollary 2. *With the hypotheses of Lemma 9.1.1, if E is an ideal sheaf $I_Z \in X^{[n]}$ and F has rank at least 2, then the radius of the corresponding wall satisfies*

$$\rho_W^2 \leq \frac{2nd + (H \cdot D)^2 - dD^2}{8d^2} := \varrho_{H,D,n}.$$

The number $\varrho_{H,D,n}$ therefore bounds the squares of the radii of higher rank walls for $X^{[n]}$.

9.2 Rank one walls and critical divisors

In the cases where we compute the Gieseker wall, the ideal sheaf that is destabilized along the wall will be destabilized by a rank 1 subobject. We first compute the numerical walls given by rank 1 subobjects.

Lemma 9.2.1. *Consider a rank 1 torsion-free sheaf $F = I_{Z'}(-L)$, where Z' is a zero-dimensional scheme of length w and L is an effective divisor. In the (H, D)-slice, the numerical wall W for $X^{[n]}$ where F has the same slope as an ideal I_Z of n points has center $(s_W, 0)$ given by*

$$s_W = -\frac{2(n - w) + L^2 + 2(D \cdot L)}{2(H \cdot L)}.$$

Proof. This is an immediate consequence of Equation (8.2) for the center of a wall. □

Recalling that walls for $X^{[n]}$ left of the vertical wall get larger as their centers decrease, we deduce the following consequence.

Lemma 9.2.2. *If the Gieseker wall in the (H, D)-slice is given by a rank 1 subobject, then it is a line bundle $\mathcal{O}_X(-L)$ for some effective divisor L.*

Proof. Suppose some $I_Z \in X^{[n]}$ is destabilized along the Gieseker wall W by a sheaf of the form $I_{Z'}(-L)$ where Z' is a nonempty zero-dimensional scheme and L is effective. By Lemma 9.2.1, the numerical wall W' given by $\mathcal{O}_X(-L)$ is strictly larger than W. Since $\mathcal{O}_X(-L)$ has the same $\mu_{H,D}$-slope as $I_{Z'}(-L)$ and $I_{Z'}(-L)$ is in the categories along W, we find that $\mathcal{O}_X(-L)$ is in at least some of the categories along W'. But then W' is an actual wall, since any ideal sheaf I_Z where Z lies on a curve $C \in |L|$ is destabilized along it. This contradicts that W is the Gieseker wall. □

Less trivially, there is a further minimality condition automatically satisfied by a line bundle $\mathcal{O}_X(-L)$ which gives the Gieseker wall. We define the set of *critical effective divisors* with respect to H and D by

$$\mathrm{CrDiv}(H, D) = \{-D\} \cup \{L \in \mathrm{Pic}(X) \text{ effective} : H \cdot L < H \cdot (-D)\}.$$

By [Har77, Ex. V.1.11], the set $\mathrm{CrDiv}(H, D)/\sim$ of critical divisors modulo numerical equivalence is finite. Therefore the set of numerical walls for $X^{[n]}$ given by line bundles $\mathcal{O}_X(-L)$ with $L \in \mathrm{CrDiv}(H, D)$ is also finite. Note that the inequality $H \cdot L < H \cdot (-D)$ is equivalent

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to the inequality $\mu_{H,D}(\mathcal{O}_X(-L)) > 0$. The next proposition demonstrates the importance of critical divisors.

Proposition 7. *Assume $2n > D^2$, and suppose the subobject giving the Gieseker wall for $X^{[n]}$ in the (H, D)-slice is a line bundle. Then the Gieseker wall is computed by $\mathcal{O}_X(-L)$, where $L \in \mathrm{CrDiv}(H, D)$ is chosen so that the numerical wall given by $\mathcal{O}_X(-L)$ is as large as possible.*

Proof. First, consider the numerical wall W given by $\mathcal{O}_X(D)$. By Lemma 9.2.1, the center $(s_W, 0)$ has

$$s_W = \frac{2n - D^2}{2(H \cdot D)} < 0 \tag{9.3}$$

since $2n > D^2$ and D is antieffective. Since $\mu_{H,D}(\mathcal{O}_X(D)) = \Delta(\mathcal{O}_X(D)) = 0$, Formula (8.3) for the radius of W gives $\rho_W^2 = s_W^2$. In particular, W is nonempty, and $\mathcal{O}_X(D)$ lies in at least some of the categories along W. Since D is antieffective, there are exact sequences of the form

$$0 \to \mathcal{O}_X(D) \to I_Z \to I_{Z \subset C} \to 0$$

where $C \in |-D|$ and $Z \subset C$ is a collection of n points. If no actual wall is larger than W, it follows that W is an actual wall and it is the Gieseker wall.

Suppose the Gieseker wall is larger than W and computed by a line bundle $\mathcal{O}_X(-L)$ with L effective. Since W passes through the origin in the (β, α)-plane, $\mathcal{O}_X(-L)$ must lie in the category $\mathcal{T}_0$. Therefore $\mu_{H,D}(\mathcal{O}_X(-L)) > 0$, and $L \in \mathrm{CrDiv}(H, D)$.

Conversely, suppose $L \in \mathrm{CrDiv}(H, D)$ is chosen to maximize the wall W' given by $\mathcal{O}_X(-L)$. Then no actual wall is larger than W'. Since $s_W < 0$ and $\mu_{H,D}(\mathcal{O}_X(-L)) \geq 0$, we find that $\mathcal{O}_X(-L)$ is in at least some of the categories along W, and hence in at least some of the categories along W'. We conclude that W' is an actual wall, and therefore that it is the Gieseker wall. $\square$

Combining Corollary 2 and Proposition 7 gives our primary tool to compute the Gieseker wall.

Theorem 14. *Assume $2n > D^2$, and let $L \in \mathrm{CrDiv}(H, D)$ be a critical divisor such that the wall for $X^{[n]}$ given by $\mathcal{O}_X(-L)$ is as large as possible. If this wall has radius ρ satisfying $\rho^2 \geq \varrho_{H,D,n}$, then it is the Gieseker wall.*

Conversely, if the Gieseker wall has radius satisfying $\rho^2 \geq \varrho_{H,D,n}$ then it is obtained in this way.

While the theorem is our sharpest result, it is useful to lose some generality to get a more explicit version. Since $-D \in \mathrm{CrDiv}(H, D)$, if the wall given by $\mathcal{O}_X(D)$ satisfies $\rho^2 \geq \varrho_{H,D,n}$ then the Gieseker wall is computed by Theorem 14. This allows us to compute the Gieseker wall so long as n is large enough, depending only on the intersection numbers of H and D.

Corollary 3. *Let*
$$\eta_{H,D} := \frac{(H \cdot D)^2 + dD^2}{2d}.$$

If $n \geq \eta_{H,D}$ then the Gieseker wall is the largest wall given by a critical divisor.

Furthermore, if $n > \eta_{H,D}$ then every I_Z destabilized along the Gieseker wall fits into an exact sequence
$$0 \to \mathcal{O}_X(-C) \to I_Z \to I_{Z \subset C} \to 0$$

for some curve $C \in |L|$, where L is a critical divisor computing the Gieseker wall. If the critical divisor computing the Gieseker wall is unique, then $\mathcal{O}_X(-C)$ and $I_{Z \subset C}$ are the Jordan–Hölder factors of any I_Z destabilized along the Gieseker wall.

Proof. Observe that the inequality $n \geq \eta_{H,D}$ automatically implies the inequality $2n > D^2$ needed to apply Theorem 14.

Let W be the wall for $X^{[n]}$ in the (H, D)-slice corresponding to $\mathcal{O}_X(D)$. The center $(s_W, 0)$ of W was computed in Equation (9.3), and $\rho_W^2 = s_W^2$. We find that $\rho_W^2 \geq \varrho_{H,D,n}$ holds when $n \geq \eta_{H,D}$, with strict inequality when $n > \eta_{H,D}$.

When $n > \eta_{H,D}$ there can be no higher-rank destabilizing subobject of an I_Z destabilized along the Gieseker wall, so there is an exact sequence as claimed. Furthermore, if there is only one critical divisor computing the wall, then there is a unique destabilizing subobject along the wall, so the Jordan–Hölder filtration has length two. □

9.3 Classes of divisors

In this subsection we give an elementary computation of the class of the divisor corresponding to a wall in a given slice of the stability manifold. Similar results have been obtained by Liu [Liu15], but the result is critical to our discussion so we include the proof. See [BM14b, §4] for more details on the definitions and results we use here.

Throughout this subsection, let $\mathbf{v} \in K_0(X)$ be a vector such that the moduli space $M_{H,D}(\mathbf{v})$ of (H, D)-Gieseker semistable sheaves admits a (quasi-)universal family $\mathcal{E}$ which is unique up to equivalence (Hilbert schemes $X^{[n]}$ are examples of such spaces). We also let $\sigma = (Z, \mathcal{A})$ be a stability condition in the closure of the Gieseker chamber for $\mathbf{v}$ in the (H, D)-slice. Then there is a well-defined corresponding divisor $D_\sigma \in N^1(M_{H,D-K_X/2}(\mathbf{v}))$ which is independent of the choice of $\mathcal{E}$.

Let $(\mathbf{v}, \mathbf{w}) = \chi(\mathbf{v} \cdot \mathbf{w})$ be the Euler pairing on $K_{\mathrm{num}}(X)_{\mathbb{R}}$, and write $\mathbf{v}^\perp \subset K_{\mathrm{num}}(X)_{\mathbb{R}}$ for the orthogonal complement with respect to this pairing. The correspondence between stability conditions and divisor classes is understood in terms of the Donaldson homomorphism

$$\lambda : \mathbf{v}^\perp \to N^1(M_{H,D-K_X/2}(\mathbf{v})).$$

Since the Euler pairing is nondegenerate, there is a unique vector $\mathbf{w}_\sigma \in \mathbf{v}^\perp$ such that

$$\Im\left(-\frac{Z(\mathbf{w}')}{Z(\mathbf{v})}\right) = (\mathbf{w}', \mathbf{w}_\sigma)$$

for all $\mathbf{w}' \in K_{\mathrm{num}}(X)_{\mathbb{R}}$. Bayer and Macrì show that $D_\sigma = \lambda(\mathbf{w}_\sigma)$. In what follows, we write vectors in $K_{\mathrm{num}}(X)_{\mathbb{R}}$ as $(\mathrm{ch}_0, \mathrm{ch}_1, \mathrm{ch}_2)$.

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Proposition 8. *With the above assumptions, suppose σ lies on a numerical wall W in the (H, D)-slice with center $(s_W, 0)$. Then $\mathbf{w}_\sigma$ is a multiple of*

$$\left(-1, -\frac{1}{2}K_X + s_W H + D, m\right) \in \mathbf{v}^\perp,$$

where m is determined by the requirement $\mathbf{w}_\sigma \in \mathbf{v}^\perp$.

In particular, if X has irregularity 0 and $\mathbf{v} = (1, 0, -n)$ is the vector for $X^{[n]}$, then the divisor D_σ is a multiple of

$$\frac{1}{2}K_X^{[n]} - s_W H^{[n]} - D^{[n]} - \frac{1}{2}B.$$

Remark 9.3.1. *Suppose X has irregularity 0. Up to scale, the divisors induced by stability conditions in the (H, D)-slice give a ray in $N^1(X^{[n]})$ emanating from the class $H^{[n]} \in \mathrm{Nef}(X) \subset \mathrm{Nef}(X^{[n]})$. The particular ray is determined by the choice of the twisting divisor D.*

Proof of Proposition 8. Since σ is in the (H, D)-slice, write $\sigma = \sigma_{\beta,\alpha}$ and $(Z, \mathcal{A}) = (Z_{\beta,\alpha}, \mathcal{A}_\beta)$ for short. Put $z = -1/Z(\mathbf{v}) = u + iv$. We evaluate the identity

$$\Im(zZ(\mathbf{w}')) = (\mathbf{w}', \mathbf{w}_\sigma)$$

defining $\mathbf{w}_\sigma$ on various classes $\mathbf{w}'$ to compute $\mathbf{w}_\sigma$.

Write the Chern character $\mathbf{w}_\sigma = (r, C, d)$. Then

$$-v = \Im(zZ(0, 0, 1)) = ((0, 0, 1), \mathbf{w}_\sigma) = r,$$

so $r = -v$. Next, for any curve class C',

$$(u + \beta v)(C' \cdot H) + v(C' \cdot D) = \Im(zZ(0, C', 0)) = ((0, C', 0), \mathbf{w}_\sigma) = \chi(0, -vC', C' \cdot C).$$

By Riemann–Roch and adjunction,

$$\chi(0, -vC', C'{\cdot}C) = -v\left(\left(-\frac{1}{v}(C' \cdot C) + \frac{1}{2}(C')^2\right) - \frac{1}{2}(C')^2 - \frac{1}{2}(C' \cdot K_X)\right) = C'{\cdot}C + \frac{v}{2}(K_X{\cdot}C'),$$

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so

$$C' \cdot C = (u + \beta v)(C' \cdot H) + v(C' \cdot D) - \frac{v}{2}(C' \cdot K_X)$$

for every class C'. Thus for any class C' with $C' \cdot H = 0$, we have $C' \cdot C = v(C' \cdot D) - \frac{v}{2}(C' \cdot K_X)$; it follows that there is some number a with

$$C = -\frac{v}{2}K_X + aH + vD.$$

Considering $C = H$ shows that $a = u + \beta v$. Therefore

$$\mathbf{w}_\sigma = (-v, -\frac{v}{2}K_X + (u + \beta v)H, m),$$

where m is chosen such that $\mathbf{w}_\sigma \in \mathbf{v}^\perp$.

Finally, a straightforward calculation shows that

$$\frac{u}{v} + \beta = \nu_\sigma(\mathbf{v}) + \beta = s_W$$

holds for all (β, α) along W. The follow up statement for Hilbert schemes follows by computing the Donaldson homomorphism. $\qquad\square$

9.4 Dual curves

Suppose D_{σ_0} is the nef divisor corresponding to the Gieseker wall for $X^{[n]}$ in the (H, D)-slice. Showing that D_{σ_0} is an extremal nef divisor amounts to showing that there is some curve $\gamma \subset X^{[n]}$ with $D_{\sigma_0} \cdot \gamma = 0$. By the Positivity Lemma, this happens when γ parameterizes objects of $X^{[n]}$ which are generically S-equivalent with respect to σ_0.

In every case where we computed the Gieseker wall, the wall can be given by a destabilizing subobject which is a line bundle $\mathcal{O}_X(-C)$ with C an effective curve. If Z is a length n subscheme of C, then there is a destabilizing sequence

$$0 \to \mathcal{O}_X(-C) \to I_Z \to I_{Z \subset C} \to 0.$$

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If $\text{ext}^1(I_{Z \subset C}, \mathcal{O}_X(-C)) \geq 2$, then curves of objects of $X^{[n]}$ which are generically S-equivalent with respect to σ_0 are obtained by varying the extension class. We obtain the following general result.

Lemma 9.4.1. *Suppose the Gieseker wall for $X^{[n]}$ in the (H, D)-slice is computed by the subobject $\mathcal{O}_X(-C)$, where C is an effective curve class of arithmetic genus $p_a(C)$. If $n \geq p_a(C) + 1$, then the corresponding nef divisor D_{σ_0} is extremal.*

Proof. Bilinearity of the Euler characteristic $\chi(\cdot, \cdot)$ and Serre duality shows that

$$\chi(I_{Z \subset C}, \mathcal{O}_X(-C)) = p_a(C) - 1 - n.$$

Therefore, once $n \geq p_a(C) + 1$ we will have $\chi(I_{Z \subset C}, \mathcal{O}_X(-C)) \leq -2$, and curves orthogonal to D_{σ_0} can be constructed by varying the extension class. $\square$

Combining Lemma 9.4.1 with our previous results on the computation of the Gieseker wall gives us the following asymptotic result.

Theorem 15. *Fix a slice (H, D) for $\text{Stab}(X)$. There is some $L \in \text{CrDiv}(H, D)$ such that for all $n \gg 0$ the Gieseker wall is computed by $\mathcal{O}_X(-L)$. Furthermore, the corresponding nef divisor is extremal.*

Proof. Recall that the set $\text{CrDiv}(H, D)/\sim$ of critical divisors modulo numerical equivalence is finite; say $\{L_1, \ldots, L_m\}$ is a set of representatives. For $1 \leq i \leq m$, let $(s_i(n), 0)$ be the center of the wall $\mathcal{O}_X(-L_i)$ for $X^{[n]}$. Then $s_i(n)$ is a linear function of n by Lemma 9.2.1, so there is some i with $s_i(n) \leq s_j(n)$ for all $1 \leq j \leq m$ and $n \gg 0$. Then by Corollary 3 the Gieseker wall is given by $\mathcal{O}_X(-L_i)$. Again increasing n if necessary, the divisor D_{σ_0} corresponding to the Gieseker wall is extremal by Lemma 9.4.1. $\square$

Remark 9.4.2. *The requirement $n \geq p_a(C) + 1$ in Lemma 9.4.1 is not typically sharp. For example, if $|C|$ contains a smooth curve we may as well assume C is smooth. Then $I_{Z \subset C}$ is*

a line bundle on C, and

$$\text{Ext}^1(I_{Z \subset C}, \mathcal{O}_X(-C)) \cong H^0(\mathcal{O}_C(Z)).$$

Thus g_n^1's on C give curves which are orthogonal to D_{σ_0}. The following fact from Brill–Noether theory therefore provides curves on $X^{[n]}$ for smaller values on n.

Lemma 9.4.3. *[ACGH85] If C is smooth of genus g, then it has a g_n^1 for any $n \geq \lceil \frac{g+2}{2} \rceil$.*

For specific surfaces, some curves in $|C|$ may have highly special linear series giving better constructions of curves on $X^{[n]}$.

Chapter 10

Picard rank one examples

For the rest of the paper, we will apply the methods of Section 9 to compute $\mathrm{Nef}(X^{[n]})$ for several interesting surfaces X. These applications form the heart of the paper.

10.1 Picard rank one in general

Suppose $\mathrm{Pic}(X) \cong \mathbb{Z}H$ for some ample divisor H. If we choose $D = -aH$, where $a > 0$ is the smallest positive integer such that aH is effective, then $\mathrm{CrDiv}(H, D) = \{-D\}$.

Lemma 10.1.1. *Suppose* $\mathrm{Pic}(X) = \mathbb{Z}H$ *and* aH *is the minimal effective class. If* $n \geq (aH)^2 = a^2 d$, *then the Gieseker wall for* $X^{[n]}$ *is the wall given by* $\mathcal{O}_X(-aH)$.

Proof. Apply Corollary 3 with $D = -aH$. □

Note that when $n > a^2 d$, additional information about the Jordan–Hölder filtration can be obtained as in Corollary 3. We use Formula (9.3) to see that the wall W given by $\mathcal{O}_X(-aH)$ has center $(s_W, 0)$ with

$$s_W = \frac{a}{2} - \frac{n}{ad}$$

Combining Lemmas 10.1.1, 9.4.1, and Proposition 8, we have proved the following general result.

Theorem 16. *Suppose* $\mathrm{Pic} X \cong \mathbb{Z} H$ *and* aH *is the minimal effective class. If* $n \geq a^2 d$ *then the divisor*

$$\frac{1}{2} K_X^{[n]} + \left(\frac{a}{2} + \frac{n}{ad} \right) H^{[n]} - \frac{1}{2} B \tag{10.1}$$

is nef. Additionally, if $n \geq p_a(aH) + 1$ *then this divisor is extremal, so* $\mathrm{Nef}(X^{[n]})$ *is spanned by this divisor and* $H^{[n]}$. *An orthogonal curve is given by letting* n *points move in a* g_n^1 *on a curve of class* aH.

Remark 10.1.2. *If* $\mathrm{Pic}(X) = \mathbb{Z} H$ *and* H *is already effective, then a different argument computes the Gieseker wall so long as* $2n > d$, *improving the bound in Lemma 10.1.1. However, fine information about the Jordan–Hölder filtration of a destabilized ideal sheaf is not obtained. In fact, if* $n \leq d$ *then the destabilizing behavior can be complicated. For instance, a scheme* Z *contained in the complete intersection of two curves of class* H *will admit an interesting map from* $\mathcal{O}_X(-H)^{\oplus 2}$.

Proposition 9. *Suppose* $\mathrm{Pic} X = \mathbb{Z} H$ *and* H *is effective. If* $2n > d$, *then the Gieseker wall for* $X^{[n]}$ *in the* $(H, -H)$-*slice is the wall given by* $\mathcal{O}_X(-H)$. *Thus the divisor (10.1) with* $a = 1$ *is nef.*

Proof. Let W be the numerical wall given by $\mathcal{O}_X(-H)$. By the proof of Proposition 7, if no actual wall is larger than W then W is an actual wall, and hence the Gieseker wall. If there is a destabilizing sequence

$$0 \to F \to I_Z \to G \to 0$$

giving a wall W' larger than W, then $F, G \in \mathcal{A}_0$ since W passes through the origin in the (β, α)-plane. Fix $\alpha > 0$ such that $(0, \alpha)$ lies on W'. We have

$$H \cdot \mathrm{ch}_1^{-H}(F) = \Im Z_{0,\alpha}(F) \geq 0 \quad \text{and} \quad H \cdot \mathrm{ch}_1^{-H}(G) = \Im Z_{0,\alpha}(G) \geq 0.$$

Since d is the smallest intersection number of H with an integral divisor and

$$d = \Im Z_{0,\alpha}(I_Z) = \Im Z_{0,\alpha}(F) + \Im Z_{0,\alpha}(G)$$

we conclude that either $\Im Z_{0,\alpha}(F) = 0$ or $\Im Z_{0,\alpha}(G) = 0$. Thus either F or G has infinite $\sigma_{0,\alpha}$-slope, contradicting that $(0, \alpha)$ is on W'. $\qquad\square$

We now further relax the lower bound on n needed to guarantee the existence of orthogonal curve classes in special cases.

10.2 Surfaces in $\mathbb{P}^3$

By the Noether–Lefschetz theorem, a very general surface $X \subset \mathbb{P}^3$ of degree $d \geq 4$ is smooth of Picard rank 1 and irregularity 0. Let H be the hyperplane class and put $D = -H$. We have $K_X = (d-4)H$, so Proposition 9 shows that if $2n > d$ then the divisor

$$\left(\frac{d}{2} - \frac{3}{2} + \frac{n}{d} \right) H^{[n]} - \frac{1}{2}B$$

is nef. If C is any smooth hyperplane section then the projection from a point on C gives a degree $d - 1$ map to $\mathbb{P}^1$, so C carries a g_n^1 for any $n \geq d - 1$. We have proved the following result.

Proposition 10. *Let X be a smooth degree d hypersurface in $\mathbb{P}^3$ with Picard rank 1. The divisor*

$$\left(\frac{d}{2} - \frac{3}{2} + \frac{n}{d} \right) H^{[n]} - \frac{1}{2}B$$

on $X^{[n]}$ is nef if $2n > d$. If $n \geq d - 1$, then it is extremal, and together with $H^{[n]}$ it spans $Nef(X^{[n]})$.

Remark 10.2.1. *The behavior of $Nef(X^{[n]})$ for smaller n in Proposition 10 is more mysterious. Even the cases $d = 5$ and $n = 2, 3$ are interesting.*

Remark 10.2.2. *The case $d = 4$ of Proposition 10 recovers a special case of [BM14b, Proposition 10.3] for K3 surfaces. The case $d = 1$ recovers the computation of the nef cone of $\mathbb{P}^{2[n]}$ [ABCH12].*

10.3 Branched covers of $\mathbb{P}^2$

Next we consider cyclic branched covers of $\mathbb{P}^2$. Let X be a very general cyclic degree d cover of $\mathbb{P}^2$, branched along a degree e curve. Note that this means that d necessarily divides e. We can view these covers as hypersurfaces in a weighted projective space, which gives us a Noether–Lefschetz type theorem: $\mathrm{Pic}X = \mathbb{Z}H$, generated by the pullback H of the hyperplane class on $\mathbb{P}^2$, provided that X has positive geometric genus. The canonical bundle of X is

$$K_X = -3H + e\left(\frac{d-1}{d}\right)H = \left(\frac{e(d-1)}{d} - 3\right)H.$$

Then X will have positive geometric genus if $e \geq 3d/(d-1)$.

Setting $D = -H$, we see that if $2n > d$ then the divisor class

$$\left(\frac{e(d-1)}{2d} - 1 + \frac{n}{d}\right)H^{[n]} - \frac{1}{2}B$$

is nef by Proposition 9. The preimage of a line is a curve of class H, and it carries a g_d^1 given by the map to $\mathbb{P}^2$. Therefore the above divisor is extremal once $n \geq d$.

Proposition 11. *Let X be a very general degree d cyclic cover of $\mathbb{P}^2$ ramified along a degree e curve, where d divides e and $e \geq \frac{3d}{d-1}$. The divisor*

$$\left(\frac{e(d-1)}{2d} - 1 + \frac{n}{d}\right)H^{[n]} - \frac{1}{2}B$$

on $X^{[n]}$ is nef if $2n > d$. For $n \geq d$, this class is extremal, and together with $H^{[n]}$ it spans $\mathrm{Nef}(X^{[n]})$.

Chapter 11

Del Pezzo surfaces of degree one

In [BC13], Bertram and Coskun studied the birational geometry of $X^{[n]}$ when X is a minimal rational surface or a del Pezzo surface. In particular, they completely computed the nef cones of all these Hilbert schemes except in the case of a del Pezzo surface of degree 1. The constructions they gave were classical: they produced nef divisors from k-very ample line bundles, and dual curves by letting collections of points move in linear pencils on special curves.

In this section, we will compute the nef cone of $X^{[n]}$, where X is a smooth del Pezzo surface of degree 1. Then $X \cong \mathrm{Bl}_{p_1,\dots,p_8} \mathbb{P}^2$ for distinct points $p_1, \dots, p_8$ with the property that $-K_X$ is ample (see [Man74, Theorem 24.4] or [?, Ex. V.21.1]). This application exhibits the full strength of the methods of Section 9.

11.1   Notation and statement of results

Let H be the class of a line and let $E_1, \dots, E_8$ be the 8 exceptional divisors over the p_i, so $\mathrm{Pic}(X) \cong \mathbb{Z}H \oplus \mathbb{Z}E_1 \oplus \cdots \oplus \mathbb{Z}E_8$ and $K_X = -3H + \sum_i E_i$. Recall that a (-1)-*curve* on X is a smooth rational curve of self-intersection -1. It is simplest to describe the dual cone of effective curves. We recommend reviewing §8.1 for notation.

Theorem 17. *The cone of curves* $\mathrm{NE}(X^{[n]})$ *is spanned by all the classes* $E_{[n]}$ *given by* (-1)-*curves* $E \subset X$, *the class of a curve contracted by the Hilbert–Chow morphism, and the class* $F_{[n]}$, *where* $F \in |-K_X|$ *is an anticanonical curve.*

The 240 (-1)-curves E on X are well-known. The possible classes are

$$(0;1) \quad (1;1^2) \quad (2;1^5) \quad (3;2,1^6) \quad (4;2^3,1^5) \quad (5;2^6,1^2) \quad (6;3,2^7),$$

where e.g. $(4;2^3,1^5)$ denotes any class equivalent to

$$4H - 2E_1 - 2E_2 - 2E_3 - E_4 - E_5 - E_6 - E_7 - E_8$$

under the natural action of S_8 on $\mathrm{Pic}(X)$. The cone of curves $\mathrm{NE}(X)$ is spanned by the classes of the (-1)-curves. The Weyl group action on $\mathrm{Pic}(X)$ acts transitively on (-1)-curve classes. It also acts transitively on systems of 8 pairwise disjoint (-1)-curves; dually, it acts transitively on the extremal rays of the nef cone $\mathrm{Nef}(X)$. We refer the reader to [Man74, §26] for details.

Consider the divisor class $(n-1)(-K_X)^{[n]} - \frac{B}{2}$. If E is any (-1)-curve on X, then $-K_X \cdot E = 1$, so

$$E_{[n]} \cdot ((n-1)(-K_X)^{[n]} - \frac{1}{2}B) = (n-1)(-K_X \cdot E) - (n-1) = 0.$$

Let $\Lambda \subset N^1(X^{[n]})$ be the cone spanned by divisors which are nonnegative on all classes $E_{[n]}$ and curves contracted by the Hilbert–Chow morphism. It follows that $\Lambda \supset \mathrm{Nef}(X^{[n]})$ is spanned by $\mathrm{Nef}(X) \subset \mathrm{Nef}(X^{[n]})$ and the single additional class $(n-1)(-K_X)^{[n]} - \frac{B}{2}$.

However, $\mathrm{Nef}(X^{[n]}) \subset \Lambda$ is a *proper* subcone. Indeed, if $F \in |-K_X|$ is an anticanonical curve then by Riemann-Hurwitz $F_{[n]} \cdot B = 2n$, so $F_{[n]} \cdot ((n-1)(-K_X)^{[n]} - \frac{B}{2}) = -1$. Let $\Lambda' \subset \Lambda$ be the subcone of $F_{[n]}$-nonnegative divisors. Taking duals, we see that Theorem 17 is equivalent to the next result.

Theorem 18. *We have* $\mathrm{Nef}(X^{[n]}) = \Lambda'$.

To prove Theorem 18, we must show that all the extremal rays of Λ' are actually nef. Suppose $N \in \text{Nef}(X)$ spans an extremal ray of $\text{Nef}(X)$. Then the cone spanned by $N^{[n]}$ and $(n-1)(-K_X)^{[n]} - \frac{B}{2}$ contains a single ray of $F_{[n]}$-orthogonal divisors, and this ray is an extremal ray of Λ'. Conversely, due to our description of the cone Λ, the extremal rays of Λ' which are not in $\text{Nef}(X)$ are all obtained in this way.

11.2 Choosing a slice

More concretely, making use of the Weyl group action we may as well assume our extremal nef class $N \in \text{Nef}(X)$ is $H - E_1$. The corresponding $F_{[n]}$-orthogonal ray described in the previous paragraph is spanned by

$$(n-1)(-K_X)^{[n]} + \frac{1}{2}(H^{[n]} - E_1^{[n]}) - \frac{1}{2}B; \tag{11.1}$$

our job is to show that this class is nef. We will prove this by exhibiting this divisor as the nef divisor on $X^{[n]}$ corresponding to the Gieseker wall for a suitable choice of slice of $\text{Stab}(X)$.

To apply the methods of Section 9, it is convenient to choose our polarization to be

$$P = \left(n - \frac{3}{2}\right)(-K_X) + \frac{1}{2}(H - E_1)$$

(which depends on n!) and our antieffective class to be $D = K_X$. Observe that P is ample since it is the sum of an ample and a nef class. If we show that the Gieseker wall W in the (P, K_X)-slice has center $(s_W, 0) = (-1, 0)$, then Proposition 8 implies the divisor class (11.1) is nef.

11.3 Critical divisors

Our plan is to apply Corollary 3 to compute the Gieseker wall in the (P, K_X)-slice. We must first identify the set $\text{CrDiv}(P, K_X)$ of critical divisors.

Lemma 11.3.1. *If $n > 2$, then the set $\mathrm{CrDiv}(P, K_X)$ consists of $-K_X$ and the classes L of (-1)-curves on X with $L \cdot (H - E_1) \leq 1$.*

When $n = 2$, the above classes are still critical. Additionally, the class $H - E_1$ is critical, as is any sum of two (-1)-curves L_1, L_2 with $L_i \cdot (H - E_1) = 0$.

Proof. Write $2P = A + N$ where $A = (2n - 3)(-K_X)$ is ample and $N = H - E_1$ is nef. Then $A \cdot (-K_X) = 2n - 3$ and $N \cdot (-K_X) = 2$, so an effective curve class $L \neq -K_X$ is in $\mathrm{CrDiv}(P, K_X)$ if and only if $L \cdot (2P) < 2n - 1$.

First suppose $n > 2$, and let $L \in \mathrm{CrDiv}(P, K_X)$. If $L \cdot (-K_X) \geq 2$, then $L \cdot (2P) \geq 4n - 6 > 2n - 1$, so L is not critical. Therefore $L \cdot (-K_X) = 1$. Thus any curve of class L is reduced and irreducible. By the Hodge index theorem,

$$L^2 = L^2 \cdot (-K_X)^2 \leq (L \cdot (-K_X))^2 = 1,$$

with equality if and only if $L = -K_X$. If the inequality is strict, then by adjunction we must have $L^2 = -1$ and L is a (-1)-curve. Since $L \cdot (2P) < 2n - 1$, we further have $L \cdot N \leq 1$.

Suppose instead that $n = 2$ and $L \in \mathrm{CrDiv}(P, K_X)$. The cases $L \cdot (2P) \leq 1$ and $L \cdot (2P) \geq 3$ follows as in the previous case. The only other possibility is that $L \cdot (-K_X) = 2$ and $L \cdot N = 0$. Since $L \cdot N = 0$, the curve L is a sum of curves in fibers of the projection $X \to \mathbb{P}^1$ given by $|N|$. This easily implies the result. $\square$

The next application of Corollary 3 completes the proof of Theorems 17 and 18.

Proposition 12. *The Gieseker wall for $X^{[n]}$ in the (P, K_X)-slice has center $(-1, 0)$, and is given by the subobject $\mathcal{O}_X(K_X)$. It coincides with the wall given by $\mathcal{O}_X(-L)$, where L is any (-1)-curve with $L \cdot (H - E_1) = 0$.*

Proof. By Equation (9.3), the center of the wall for $\mathcal{O}_X(K_X)$ is $(s_W, 0)$ with

$$s_W = \frac{2n - K_X^2}{(2P) \cdot K_X} = -1.$$

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A straightforward computation shows $\eta_{P,K_X} < n$ for all $n \geq 2$. Therefore, by Corollary 3, the Gieseker wall is computed by a critical divisor.

We only need to verify that no other critical divisor gives a larger wall. Let $L \in \mathrm{CrDiv}(P, K_X)$. By Lemma 9.2.1, the center of the wall given by $\mathcal{O}_X(-L)$ lies at the point $(s_L, 0)$ where

$$s_L = -\frac{2n + L^2 + 2(K_X \cdot L)}{(2P) \cdot L}.$$

If L is a (-1)-curve, then

$$s_L = -\frac{2n-3}{(2P) \cdot L} = -\frac{2n-3}{2n-3+L \cdot (H-E_1)} \geq -1,$$

with equality if and only if $L \cdot (H - E_1) = 0$. This proves the result if $n > 2$.

To complete the proof when $n = 2$, we only need to consider the additional critical classes mentioned in Lemma 11.3.1. For every such $L \in \mathrm{CrDiv}(P, K_X)$ we have $L \cdot K_X = -2$ and $L^2 \leq 0$. Thus $s_L \geq 0$ for every such divisor. $\qquad \square$

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