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# Condizioni di stabilità di Bridgeland su varietà con canonico banale e trasformate di Fourier-Mukai

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*"E tenebris tantis tam clarum extollere lumen*
*qui primus potuisti inlustrans commoda vitae*
*te saequor, o Graiae gentis decus, inque tuis nunc*
*ficta pedum pono pressis vestigia signis*
*non ita certandi cupidus, quam propter amorem,*
*quod te imitari aveo; quid enim contendat hirundo*
*cycnis, aut quid nam tremulis facere artubus haedi*
*concsimile in cursu possint et fortis equi vis?*
*Tu, pater, es rerum inventor, tu patria nobis*
*suppeditas praecepta, tuisque ex, inclute, chartis,*
*floriferis ut apes in saltibus omnia libant,*
*omnia nos itidem depascimur aurea dicta,*
*aurea, perpetua semper dignissima vita."*

Lucrezio, De Rerum Natura
Libro III, vv 1-30

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# Introduction

The idea of triangulated category and the one of derived category, which are originally due to Grothendieck and Verdier, have been at the center of a good portion of contemporary research in algebraic geometry.
The axioms behind the notion of derived category seem, at a first glance, somewhat unnatural. To answer the question: "why do we need to study derived categories?" it is interesting to read Verdier's words in his PhD thesis.

*"Dans l' étude des foncteurs dérivés d'un foncteur composé de deux foncteurs, des propriétés d'associativité, des relations du type de Künneth, on est amené à étendre le formalisme des foncteurs dérivés au cas où l'argument n'est plus seulement un objet de la catégorie étudiée, mais un complexe d'objets de cette catégorie. "*

Hence, the most natural answer to the question "why derived categories?" lies in the world of derived functors.
In algebraic geometry, there exists a class of functors which arise in a natural way (the global sections of a coherent sheaf, the need to pull-back or the push forward of a coherent sheaf, the tensor product of sheaves, etc.). These are functors from the category Coh to itself. The category Coh is abelian and it makes sense to talk about exact sequences in Coh and ask wether these functors are exact. Some of these functors are just right or left exact an the lack of exactness is in some sense measured by the derived functors of the given functor.
It turns out that the most natural context to define these functors is not the category Coh itself but its derived category in which the objects are complexes of coherent sheaves rather than coherent sheaves.
The structure of a triangulated category comes up while studying derived categories. What happens is that the derived category of an abelian category is in general not abelian, but it admits, nonetheless, an additional structure which makes it behave almost like an abelian category. A triangulated category axiomatizes this behavior. In a triangulated category exact sequences are replaced by *distinguished triangles*.
Given the derived category of a variety $\mathscr{D}(X)$, it is natural to consider its group of exact autoequivalences $\mathrm{Aut}\mathscr{D}(X)$. A powerful tool, in this context, is the *Fourier-Mukai tranform*: the idea behind it is that one can go from the derived category of a certain variety $X$ to the derived category of another varitety $Y$ via the derived category of their product, $X \times Y$. To each object of the derived category $\mathscr{E}^\bullet \in \mathscr{D}(X \times Y)$ one can associate a Fourier-Mukai transform $\Phi_{\mathscr{E}^\bullet}$ and give explicit condition for it to be fully faithful. The importance of Fourier-Mukai transforms lies in a theorem due to Orlov, namely Theorem 3.1.5: given an exact autoequivalence of $\mathscr{D}(X)$, there exists an object in $\mathscr{D}(X \times X)$ such that the corresponding Fourier-Mukai

transform is isomorphic to the exact autoequivalence itself. This means that the group $\mathrm{Aut}\mathscr{D}(X)$ can be described in terms of the category $\mathscr{D}(X \times X)$. This will be very useful when considering the natural action of the group $\mathrm{Aut}\mathscr{D}(X)$ on the variety of Bridgeland's stability conditions.
In fact the main objective of this thesis, is to study Bridgeland's idea of extending the notion of a stability condition to an arbitrary triangulated category.
The notion of stability first came up in building moduli spaces of sheaves on a fixed variety. Given an ample divisor $H$ on a complex variety $X$, one can define the *slope* of a torsion - free sheaf $\mathscr{F}$ with respect to $H$ as:

$$\mu_H(\mathscr{F}) := \frac{c_1(\mathscr{F}) \cdot H}{\mathrm{rk}(\mathscr{F})};$$

a sheaf $\mathscr{F}$ is called *stable* if for each subsheaf $\mathscr{E} \subset \mathscr{F}$, one has $\mu_H(\mathscr{E}) < \mu_H(\mathscr{F})$ or, equivalently, for each quotient $\mathscr{F} \longrightarrow \mathscr{G}$ one has $\mu_H(\mathscr{F}) < \mu_H(\mathscr{G})$.
The idea behind Bridgekland's theory has been first introduced by Douglas, who developed the theory of $\Pi$-stability in the context of D-branes, then it was generalized by Bridgeland in [2]. What Bridgeland observed is that to get some nicely behaved stability condition, one has to require a positivity condition on nonzero objects. He defines a *stability function* on al abelian category $\mathscr{A}$ as a group homomorphism

$$Z : K(\mathscr{A}) \longrightarrow \mathbb{C}$$

such that if $[E] \neq [0]$ in the Grothendieck group $K(\mathscr{A})$, then $Z([E])$ lies in the positive upper-half plane $\mathbb{H} = \{re^{i\pi\phi} \mid \phi \in (0, 1]\}$. Then the slope of an object $E \in \mathscr{A}$ is defined as

$$\phi(E) = -\frac{\Re Z([E])}{\Im Z([E])},$$

where $[E]$ is the class of $E$ in the Grothendieck group. Exactly as in the old notion of stability, an object is called *semistable* (resp. *stable* ) if for each proper subobject $F \subset E$, one has $\phi(F) \leq \phi(E)$ (resp $\phi(F) < \phi(E)$). Roughly speaking, one is asking that the ordering given by phases behaves well with respect to subobjects and quotients, i.e. with respect to the abelian character of the category $\mathscr{A}$. Everything goes well, until one tries to deform a stability function (i.e., considering small perturbation of the phases of semistable objects ). The problem, in fact, is that there are stability function which, when deformed, do not respect the positivity condition anymore: intuitively, one can think of the case when there exist semistable objects with phase equal to one. The problem is then solved by considering a triangulated category, instead of an abelian one. This idea simply comes by noticing that if a vector lies in the lower half plane, then its image via the reflection with respect to the origin lies in the upper-half plane again, and from the fact that, in the Grothendieck group of a triangulated category $\mathscr{D}$, one has that $-[E] = [E[1]]$ for each $[E] \in K(\mathscr{D})$. Therefore, we need to shift object in order to preserve the positivity condition. Unfortunately, when passing from an abelian to a triangulated category, one loses the concepts of "subobject" and "quotient": a new tool is needed in order to take the ordering back. Rougly speaking, what one does is to *slice* the

category into slices which are indexed by real numbers, so that it becomes again possible to talk about phases.
The next step is to take the set $\mathrm{Stab}(\mathscr{D})$ of all the stability conditions on a triangulated category $\mathscr{D}$ and put a topology on it. A key result (Theorem 2.4.8) is that the topological space $\mathrm{Stab}(\mathscr{D})$ can be endowed with the structure of a complex manifold. To study this space is one can study all the moduli spaces of objects in the derived category $\mathscr{D}$ which can be built by varying the stability condition. In particular, we will fix our attention to the case when $\mathscr{D}$ is the derived category of a K3 surface $X$. The space $\mathrm{Stab}(X) := \mathrm{Stab}(\mathscr{D}^b(X))$ has a wall-and-chamber structure. This means that the condition for an object to be stable is preserved through small deformations of the stability condition, until one crosses one of the walls (which will be codimension-one subvarieties), which marks a ”border” of stability condition. Then, the question one can ask is: "what happens to the moduli space of objects in the category when crossing a wall?". The answer is that there are particular birationalities, called *Mukai flops*, which map a given moduli space to the ones across the various walls.

# Chapter 1

# Triangulated and derived categories

## 1.1 Triangulated categories

Triangulated categories were first introduced by Verdier in his PhD thesis, whose goal was to supply to the structure of an abelian category whenever it was not available. The rise of this additional structure came up while studying the derived category of an arbitrary abelian category: the point is that, even if a category $\mathscr{A}$ is abelian, its derived category $\mathcal{D}(\mathscr{A})$ is almost never abelian (we will later produce a very convincing example of it). The peculiarity of a triangulated structure is that the objects characterizing abelian categories, i.e. short exact sequences, are replaced by similar objects, the so called *distinct triangles*, which in some way "behave like" short exact sequences. This will allow us to define a weaker version of the concepts of "subobject" and "quotient", so that it will be easier to replicate what we usually do in abelian categories. Let us start recalling a definition:

**Definition 1.1.1.** Let $\mathscr{C}$, $\mathscr{D}$ be additive categories. An *additive equivalence* between $\mathscr{C}$ and $\mathscr{D}$ is an equivalence $F : \mathscr{C} \to \mathscr{D}$ so that, for every couple of objects $A, B \in \mathscr{C}$, the function $\mathrm{Hom}_{\mathscr{C}}(A, B) \xrightarrow{\cong} \mathrm{Hom}_{\mathscr{D}}(F(A), F(B))$ (I will usually write $\mathrm{Hom}(\bullet, \bullet)$ omitting the ambient category when it is beyond misunderstanding) is an isomorphism of abelian groups.

We are now ready to define a triangulated structure on an additive category.

**Definition 1.1.2.** Let $\mathscr{D}$ be an additive category. A *triangulated structure* on $\mathscr{D}$ is given by:

- an additive equivalence $T : \mathscr{D} \longrightarrow \mathscr{D}$, called *shift functor*;
- a class of *distinguished triangles* (shortly DT ), i.e. triangles of the form $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} T(A)$, with $A, B, C \in \mathrm{Ob}(\mathscr{D})$, verifying the following axioms:

**TC1** **i)** Every triangle of the form $A \xrightarrow{id} A \to 0 \to T(A)$ is a DT;

**ii)** If $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} T(A)$ is a DT, then its shift $B \xrightarrow{g} C \xrightarrow{h} T(A) \xrightarrow{T(f)}$ is a DT;

**iii)** Any morphism $A \stackrel{f}{\longrightarrow} B$ can be completed to a DT: it means that there exists an object $C$ and a morphism $B \stackrel{g}{\longrightarrow} C$ so that the triangle $A \stackrel{f}{\longrightarrow} B \stackrel{g}{\longrightarrow} C \stackrel{h}{\longrightarrow} T(A)$ is a DT;

A morphism of DT is a commuting diagram of the form:

$$\begin{array}{ccccccc} A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & T(A) \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle T(\alpha)} \\ A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & T(A'). \end{array}$$

A morphism of DT is an isomorphism if $\alpha, \beta, \gamma$ are isomorphism in $\mathscr{D}$ (note that if $\alpha$ is an isomorphism, then $T(\alpha)$ is an isomorphism, too, because the functor $T$ is an equivalence).

**TC2** Any triangle isomorphic to a DT is a DT itself.

**TC3** If $A \longrightarrow B \longrightarrow C \longrightarrow T(A)$ and $A' \longrightarrow B' \longrightarrow C' \longrightarrow T(A')$ are DT and if there exists a pair of morphisms $\alpha : A \longrightarrow A'$ and $\beta : B \longrightarrow B'$ so that the square

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} \\ A' & \longrightarrow & B' \end{array}$$

commutes, then there exists a morphism $\gamma : C \longrightarrow C'$ (not necessarily unique) so that the diagram

$$\begin{array}{ccccccc} A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & T(A) \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \vdots{\scriptstyle \gamma} & & \downarrow{\scriptstyle T(\alpha)} \\ A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & T(A') \end{array}$$

commutes.

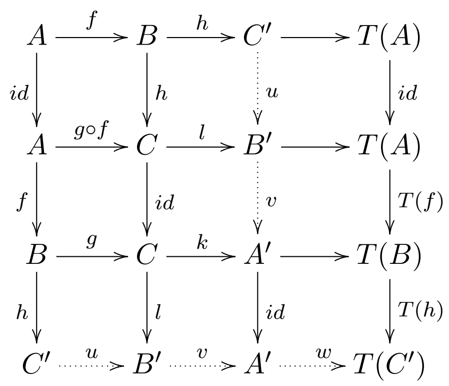
**TC4** (**Octahedron axiom**) given the following three DTs:

$$A \stackrel{f}{\longrightarrow} B \stackrel{h}{\longrightarrow} C' \longrightarrow T(A)$$

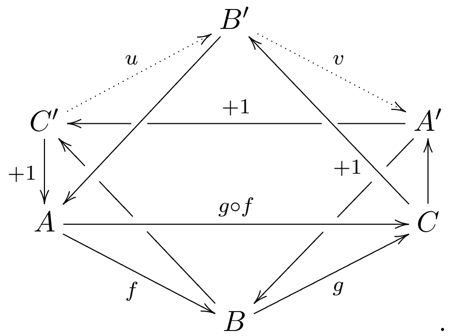
$$B \stackrel{g}{\longrightarrow} C \stackrel{k}{\longrightarrow} A' \longrightarrow T(B)$$

$$A \stackrel{g \circ f}{\longrightarrow} C \stackrel{l}{\longrightarrow} B' \longrightarrow T(A)$$

there exist a DT $C' \stackrel{u}{\longrightarrow} B' \stackrel{v}{\longrightarrow} A' \stackrel{w}{\longrightarrow} T(C')$ so that the diagram below commutes:




this diagram can also be displayed as an octahedron (hence the name):

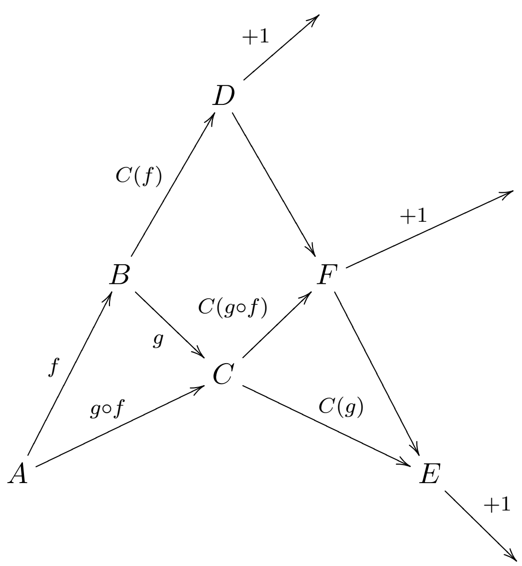



.

Sometimes it could be useful to visualize the octahedron axiom as follows: if we have a composition

$$A \xrightarrow{f} B \xrightarrow{g} C$$

then the cones of $f, g$ and $g \circ f$ fit in a distinguished triangle:



Roughly speaking, we set a class of object resembling short exact sequence, which we ask to contain trivial objects, to have enough objects and morphisms between them, and to be closed under shift and isomorphism.

**Notation 1.1.3.** We will usually write $A[1]$ and $f[1]$ for $T(A)$ and $T(f)$ respectively, and a DT might also be written as $A \longrightarrow B \longrightarrow C \xrightarrow{+1}$.

There follow some ready-made properties:

**Properties:**
We recall that, if $(\mathscr{D}, T)$ is a triangulated category and $\mathscr{A}$ and abelian category, an additive functor $F : \mathscr{D} \longrightarrow \mathscr{A}$ is *cohomological* if whenever $A \longrightarrow B \longrightarrow C \xrightarrow{+1}$ is a DT in $\mathscr{D}$, then $F(A) \longrightarrow F(B) \longrightarrow F(C)$ is an exact sequence in $\mathscr{A}$.

1. *The composition of two subsequent arrows in a DT is zero.*

   *Proof.* There is a commutative diagram:

$$\begin{array}{ccccccc} A & \xrightarrow{id} & A & \longrightarrow & 0 & \longrightarrow & T(A) \\ \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle f} & & \vdots & & \downarrow{\scriptstyle id} \\ A & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow & T(A). \end{array}$$

   The two triangles are both distinct and the first square on the left obviously commutes, so there exists an arrow (the dotted one in the diagram, which is clearly trivial), so that everything commutes. In particular, we have $g \circ f = f \circ 0 = 0$. □

2. *For any $X \in \mathscr{D}$ the functors* $\operatorname{Hom}(X, \bullet)$ *and* $\operatorname{Hom}(\bullet, X)$ *are cohomological functors between $\mathscr{D}$ and **Ab**, where **Ab** is the category af abelian groups.* (Note that **TC1 ii)** implies that each DT gives rise to a long exact sequence in **Ab**).

   *Proof.* We want to prove that if $A \longrightarrow B \longrightarrow C \xrightarrow{+1}$ the sequences

$$\operatorname{Hom}(X, A) \longrightarrow \operatorname{Hom}(X, B) \longrightarrow \operatorname{Hom}(X, C)$$

$$\operatorname{Hom}(C, X) \longrightarrow \operatorname{Hom}(B, X) \longrightarrow \operatorname{Hom}(A, X)$$

   are exact. Let us consider the first sequence: the fact that $\operatorname{Ker}\beta \supset \operatorname{Im}\alpha$ follows from the first property. To prove the converse, consider $\psi \in \operatorname{Hom}(X, B)$ such that $g \circ \psi = 0$. We have the following diagram:

$$\begin{array}{ccccccc} X & \xrightarrow{Id} & X & \longrightarrow & 0 & \longrightarrow & X[1] \\ \vdots & & \downarrow{\scriptstyle \psi} & & \downarrow & & \vdots \\ A & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow & A[1] \end{array}$$

the center square commutes, so we can applying a slightly modified third axiom to get the dotted map which lifts $\psi$. Arguing in a similar way, one proves that the second sequence is exact, too.

□

3. *If we have a morphism between two triangles*

$$\begin{array}{ccccccc} A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & A[1] \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\alpha[1]} \\ A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & A'[1]. \end{array}$$

*and any two vertical arrows are isomorphisms, so is the third one (nice use of the five lemma).*

*Proof.* We apply the first of all the third axiom to get an arrow $C \longrightarrow C'$ which makes everything commute, then we use the preceding property. Let us suppose that the first two of the vertical arrows are isomorphisms. Applying the functor $\mathrm{Hom}(X, \bullet)$ to the diagram, we get:

$$\begin{array}{ccccccccc} \mathrm{Hom}(X,A) & \longrightarrow & \mathrm{Hom}(X,B) & \longrightarrow & \mathrm{Hom}(X,C) & \longrightarrow & \mathrm{Hom}(X,A[1]) & \longrightarrow & \mathrm{Hom}(X,B[1]) \\ {\scriptstyle\mathrm{Hom}(X,\alpha)}\downarrow\cong & & {\scriptstyle\mathrm{Hom}(X,\beta)}\downarrow\cong & & {\scriptstyle\mathrm{Hom}(X,\gamma)}\downarrow & & {\scriptstyle\mathrm{Hom}(X,\alpha[1])}\downarrow\cong & & {\scriptstyle\mathrm{Hom}(X,\beta[1])}\downarrow\cong \\ \mathrm{Hom}(X,A') & \longrightarrow & \mathrm{Hom}(X,B') & \longrightarrow & \mathrm{Hom}(X,C') & \longrightarrow & \mathrm{Hom}(X,A'[1]) & \longrightarrow & \mathrm{Hom}(X,B'[1]) \end{array}$$

where, for all $X \in \mathscr{D}$, all the vertical arrows except the middle one are isomorphisms, and everything commutes because it commutes first of all in the two TDs diagram. Then, the five lemma ensure us that the middle one, too, is an isomorphism. But the fact that $\mathrm{Hom}(X,C) \xrightarrow{\cong} \mathrm{Hom}(X,C')$ is an isomorphism for all $X \in \mathscr{D}$ implies that $C \xrightarrow{\cong} C'$ is an isomorphism, too.

□

4. *The direct sum*[1] *of two DTs is a DT.*

*Proof.* Let $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{+1}$ and $A' \xrightarrow{f'} B' \xrightarrow{g'} C' \xrightarrow{+1}$ be distinct triangle. The morphism $A \oplus A' \xrightarrow{(f,f')} B \oplus B'$ can be completed to a distinct triangle: $A \oplus A' \xrightarrow{(f,f')} B \oplus B' \xrightarrow{\alpha} D \xrightarrow{\beta} A[1] \oplus A'[1]$[2]. We have two diagrams:

$$\begin{array}{ccccccc} A & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow & A[1] \\ \downarrow{\scriptstyle\begin{pmatrix} Id_A \\ 0 \end{pmatrix}} & & \downarrow{\scriptstyle\begin{pmatrix} Id_B \\ 0 \end{pmatrix}} & & \downarrow{\scriptstyle\varphi} & & \downarrow \\ A \oplus A' & \xrightarrow[(f,f')]{} & B \oplus B' & \longrightarrow & D & \longrightarrow & A[1] \oplus A'[1] \end{array}$$

[1]To be precise, here the symbol of direct sum means the coproduct in $\mathscr{D}$, which is first of all additive.

[2]Notice that $(A \oplus A')[1] \cong A[1] \oplus A'[1]$ because the shift functor is an equivalence.

$$
\begin{array}{ccccccc}
A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' & \longrightarrow & A[1] \\
\left(\begin{smallmatrix} 0 \\ Id_{A'} \end{smallmatrix}\right)\Big\downarrow & & \left(\begin{smallmatrix} 0 \\ Id_{B'} \end{smallmatrix}\right)\Big\downarrow & & \Big\downarrow \varphi' & & \Big\downarrow \\
A \oplus A' & \xrightarrow[(f,f')]{} & B \oplus B' & \longrightarrow & D & \longrightarrow & A[1] \oplus A'[1]
\end{array}
$$

which are both commutative because of the third axiom. Therefore, also the following one is commutative:

$$
\begin{array}{ccccccc}
A \oplus A' & \longrightarrow & B \oplus B' & \longrightarrow & C \oplus C' & \longrightarrow & A[1] \oplus A'[1] \\
Id_{A \oplus A'}\Big\downarrow & & \Big\downarrow Id_{B \oplus B'} & & \Big\downarrow (\varphi,\varphi') & & \Big\downarrow \\
A \oplus A' & \longrightarrow & B \oplus B' & \xrightarrow{\alpha} & D & \xrightarrow{\beta} & A[1] \oplus A'[1].
\end{array}
$$

We have thus found a map $C \oplus C' \overset{(\varphi,\varphi')}{\longrightarrow} D$ which makes everything commute. We just need to show that this map is an isomprphism. Apply the functor $\mathrm{Hom}(X, \bullet)$ to the diagram:

$$
\begin{array}{ccccccccc}
\mathrm{Hom}(X, A \oplus A') & \to & \mathrm{Hom}(X, B \oplus B') & \to & \mathrm{Hom}(X, C \oplus C') & \to & \mathrm{Hom}(X, A[1 \oplus A'[1]) & \to & \mathrm{Hom}(X, B[1] \oplus B'[1]) \\
\Big\downarrow \mathrm{Hom}(X, Id_{A \oplus A'}) & & \Big\downarrow \mathrm{Hom}(X, Id_{B \oplus B'}) & & \Big\downarrow \mathrm{Hom}(X, (\varphi,\varphi')) & & \Big\downarrow & & \Big\downarrow \\
\mathrm{Hom}(X, A \oplus A') & \to & \mathrm{Hom}(X, B \oplus B') & \longrightarrow & \mathrm{Hom}(X, D) & \longrightarrow & \mathrm{Hom}(X, A[1] \oplus A'[1]) & \to & \mathrm{Hom}(X, B[1] \oplus B'[1])
\end{array}
$$

the first row is exact because it is the direct sum of two exact sequences (remember that $\mathrm{Hom}(X, Y \oplus Z) \cong \mathrm{Hom}(X, Y) \oplus \mathrm{Hom}(X, Z)$ for all $X, Y, Z \in \mathscr{D}$), while the second one is exact because we are just applying the functor $\mathrm{Hom}(X, \bullet)$ to a DT. Moreover, all the vertical arrows except the middle one are isomorphisms (again, $\mathrm{Hom}(X, \bullet)$ applied to the identity map), so the middle one is an isomorphism, too, because of the five lemma. But, by the arbitrary choice of $X$, we can conclude that the map $C \oplus C' \overset{(\varphi,\varphi')}{\longrightarrow} D$ is an isomorphism, as well.

$\square$

5. *If* $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} A[1]$ *is a DT and* $h$ *is the zero morphism, then* $B \cong A \oplus C$.

*Proof.* The triangle $A \overset{(Id_A,0)}{\longrightarrow} A \oplus C \overset{(0,Id_C)}{\longrightarrow} C \xrightarrow{h} A[1]$ is the direct sum of the triangles $A \xrightarrow{Id_A} A \longrightarrow 0 \longrightarrow A[1]$ and $0 \longrightarrow C \xrightarrow{Id_C} C \longrightarrow 0$ which are distinct because of axiom TC1, so it is distinct itself. Therefore, applying the functor $\mathrm{Hom}(X, \bullet)$ for some $X \in \mathscr{D}$, we get an exact sequence:

$$
\mathrm{Hom}(X, A) \longrightarrow \mathrm{Hom}(X, A) \oplus \mathrm{Hom}(X, C) \longrightarrow \mathrm{Hom}(X, C) \overset{\mathrm{Hom}(X,h)}{\longrightarrow} \mathrm{Hom}(X, A[1])\,.
$$

By exactness, the map $\mathrm{Hom}(X, h)$ must be zero, and by the fact that we can choose an arbitrary $X \in \mathscr{D}$, we get that $h = 0$ itself. For the converse, let us consider the following diagram:

$$\begin{array}{ccccccc}
C[-1] & \xrightarrow{h[-1]} & A & \longrightarrow & B & \longrightarrow & C \\
\downarrow{\scriptstyle Id} & & \downarrow{\scriptstyle Id} & & \vdots & & \downarrow{\scriptstyle Id} \\
C[-1] & \xrightarrow{0} & A & \longrightarrow & A \oplus C & \longrightarrow & C
\end{array}$$

we supposed $h = 0$, so $h[-1] = 0$, too, because the shift functor is an equivalence. The first square obviously commutes, so the dotted arrow in the diagram exists and everything commutes. Moreover, we have two out of the three vertical arrows which are isomorphisms, so applying property 2. we get that the third one is an isomorphism, too.

□

6. *If* $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} T(A)$ *is a DT, then* $A \cong B$ *iff* $C \cong 0$.

*Proof.* As usual, let us apply the functor $\mathrm{Hom}(X, \bullet)$ to the DT:

$$\mathrm{Hom}(X, C[-1]) \longrightarrow \mathrm{Hom}(X, A) \longrightarrow \mathrm{Hom}(X, B) \longrightarrow \mathrm{Hom}(X, C) \longrightarrow \mathrm{Hom}(X, A[1])\ .$$

Now, $C \cong 0$ (and therefore $C[-1] \cong 0$ because the inverse of the shift functor is an equivalence, as well) if and only if $\mathrm{Hom}(X, C) = 0$ and $\mathrm{Hom}(X, C[-1]) = 0$ for all $X \in \mathscr{D}$ and, in this case, $\mathrm{Hom}(X, A) \cong \mathrm{Hom}(X, B)$. But this holds for all $X \in \mathscr{D}$ if and only if $A \cong B$. □

We will now prove a proposition which will help us later, when our goal will be to prove that the derived category of an abelian category in general is not abelian.

**Proposition 1.1.4.** *Let $\mathscr{A}$ be an abelian triangulated category. Then $\mathscr{A}$ is semisimple, i.e. any short exact sequence in $\mathscr{A}$ splits.*

*Proof.* Let $0 \longrightarrow A \xrightarrow{f} B$ be a monomorphism in $\mathscr{A}$. The morphism $A \xrightarrow{f} B$ can be completed to a DT , i.e. there exist an object $C \in \mathscr{D}$ and a morphism $B \xrightarrow{g} C$ so that the triangle $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} A[1]$ is distinguished. It means that the triangle $C[-1] \xrightarrow{h[-1]} A \xrightarrow{f} B \xrightarrow{g} C$ is distinguished, too. By property **1)** we get that $h[-1] \circ f = 0$ and the fact that f is a monomorphism implies $h[-1] = 0$. But the shift functor (and therefore its inverse) is an equivalence, so $h = 0$. By property **4)**, this means that $B \cong A \oplus C$ □

## 1.2 Derived categories: a rough construction

We will now give an idea of what the derived category of an abelian category is. Note that almost everything we are going to introduce in this section can be done with arbitrary caregories, but we will treat just the abelian case, because we are mainly interested in it. Let $\mathscr{A}$ be an abelian category, as usual. We are now interested in the *complexes* of objects which belong to $\mathscr{A}$.

**Definition 1.2.1.** A *complex* in $\mathscr{A}$ is a diagram of this form:

$$... \longrightarrow A^{i-1} \stackrel{d^{i-1}}{\longrightarrow} A^i \stackrel{d^i}{\longrightarrow} A^{i+1} \stackrel{d^{i+1}}{\longrightarrow} ... \,,$$

where:

1. $A^j \in \mathscr{A}$,
2. $d^{j+1} \circ d^j = 0$

for all $j \in \mathbb{Z}$.

**Notation 1.2.2.** We will usually write $A^\bullet$ instead of $(... \longrightarrow A^{i-1} \stackrel{d^{i-1}}{\longrightarrow} A^i \stackrel{d^i}{\longrightarrow} A^{i+1} \stackrel{d^{i+1}}{\longrightarrow} ...)$.

**Definition 1.2.3.** A morphism of complexes $f^\bullet : A^\bullet \longrightarrow B^\bullet$ is a family of arrows $\{f^j\}_{j\in\mathbb{Z}}$ such that the diagram

$$\begin{array}{ccccccccc} ... & \longrightarrow & A^{i-1} & \xrightarrow{d_A^{i-1}} & A^i & \xrightarrow{d_A^i} & A^{i+1} & \xrightarrow{d_A^{i+1}} & ... \\ & & \downarrow{\scriptstyle f^{i-1}} & & \downarrow{\scriptstyle f^i} & & \downarrow{\scriptstyle f^{i+1}} & & \\ ... & \longrightarrow & B^{i-1} & \xrightarrow{d_B^{i-1}} & B^i & \xrightarrow{d_B^i} & B^{i+1} & \xrightarrow{d_B^{i+1}} & ... \end{array}$$

commutes, i.e. $d_A^{j+1} \circ f^{j+1} = f^j \circ d_B^j \; \forall j \in \mathbb{Z}$.

It is easy to show that the class of morphisms contains identity and is closed under composition. Therefore we can conclude:

**Definition 1.2.4.** The complexes of objects in $\mathscr{A}$ together with the morphisms of complexes defined as above form a category, which we will call $Kom(\mathscr{A})$ .

A quick result is that the category of complexes of an abelian category is abelian, too. Given a morphism $f^\bullet : A^\bullet \longrightarrow B^\bullet$, its kernel can be defined as the complex $(\mathrm{Ker} f^\bullet)^i = \mathrm{Ker}(f^i)$ (cokernel will be defined in an analogous way). An easy check shows then that for every morphism $f^\bullet$ as above, there is an isomorphism $\mathrm{Ker} f^\bullet \cong \mathrm{Coker} f^\bullet$. Although, we are not interested in working with $Kom(\mathscr{A})$: the problem is, we want to identify complexes with the same cohomology. We recall that the cohomology of a complex $A^\bullet$ is:

$$H^i(A^\bullet) = \frac{\mathrm{Ker} d_A^i}{\mathrm{Im} d_A^{i-1}} \;.$$

The cohomology of the complex $0^\bullet = (... \longrightarrow 0 \longrightarrow ... \longrightarrow 0 \longrightarrow ...)$ is obviously zero for all integers, but the converse is not true: $H^i(A^\bullet) = 0 \;\; \forall i \in \mathbb{Z}$ simply means that the complex $A^\bullet$ is *exact*, i.e. $\mathrm{Ker} d_A^i = \mathrm{Im} d_A^{i-1} \;\; \forall i \in \mathbb{Z}$. Derived categories are built in order to avoid this. Let us give a definition:

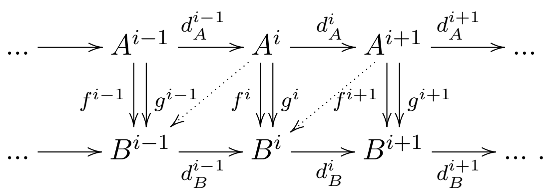
**Definition 1.2.5.** Let $f^\bullet : A^\bullet \longrightarrow B^\bullet$ be a morphism of complexes. Then $f^\bullet$ is a *quasi-isomorphism* (or, shortly, *qis*) if $H^i(f^\bullet) : H^i(A^\bullet) \xrightarrow{\cong} H^i(B^\bullet)$ is an isomorphism for all integers $i$.

What want to do is to invert formally qis's or, in other words, *localize* the class of morphisms with respect to the subclasses of qis's. The process, though, is very long and detailed, so we will just give the idea of what morphisms are. Let's first give an intermediate step, which will make everything work: to define the derived category of $\mathscr{A}$, we need to pass through the so-called *homotopy category* $K(\mathscr{A})$.

**Definition 1.2.6.** Let $f^\bullet, g^\bullet : A^\bullet \longrightarrow B^\bullet$ be morphisms of complexes. Then $f^\bullet$ and $g^\bullet$ are *homotopically equivalent* if for each $i \in \mathbb{Z}$ there exists a map $h^i : A^i \longrightarrow B^{i-1}$ so that

$$f^i - g^i = h^{i+1} \circ d_A^i + d_B^{i-1} \circ h^i \ ,$$

as shown in the following diagram:

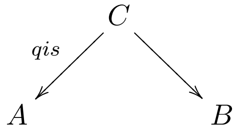



It's easy to check homotopic equivalence is an equivalence relation.

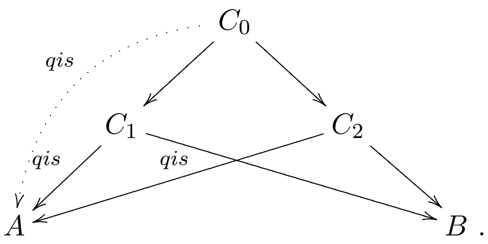
So we can define the homotopy category in this way:

- $\mathrm{Ob}(\mathscr{K}(\mathscr{A})) = \mathrm{Ob}(Kom(\mathscr{A}))$,
- $\forall A, B \in \mathscr{K}(\mathscr{A}) \ , \ \mathrm{Hom}_{\mathscr{K}(\mathscr{A})}(A,B) = \mathrm{Hom}_{Kom(\mathscr{A})}(A,B)/\sim$

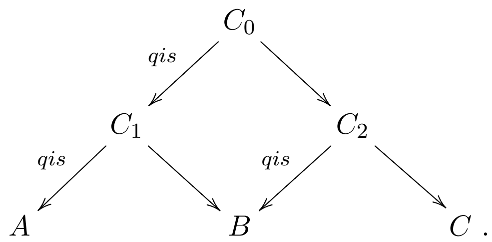
where $\sim$ is homotopic equivalence. We need to pass through homotopy category because we will ask any of the diagrams we are going to introduce to commute up to homotopy. Let us now define the derived category. The objects of $D(\mathscr{A})$ are exactly the same objects of $Kom(\mathscr{A})$ and $\mathscr{K}(\mathscr{A})$, i.e. the complexes of $\mathscr{A}$. To describe the class of morphism, we need to consider that, if qis's are to be considered as isomorphisms, any morphism $A^\bullet \longrightarrow B^\bullet$ should count as a morphism $C^\bullet \longrightarrow B^\bullet$, if $C^\bullet \longrightarrow A^\bullet$ is a qis. Therefore, given any two objects $A^\bullet, B^\bullet \in \mathscr{D}(\mathscr{A})$, a morphism between $A^\bullet$ and $B^\bullet$ is an equivalence class of diagrams of this type:




where two such diagrams are equivalent if there exists a third commutative (note: in $\mathscr{K}(\mathscr{A})$!) diagram of this type:




It is easy to check that the one defined above is actually an equivalence relation. Now we need to define the composition of two morphisms, and verify that identity is a morphism, and the composition is associative. Given two morphisms, between respectively $A^\bullet$, $B^\bullet$ and $C^\bullet$ in $\mathscr{A}$, the composition is defined as a diagram:




Associativity comes from the fact that everything commutes in $\mathscr{K}(\mathscr{A})$, while it is obvious that identity is a morphism (take $B = A$, and $C$ quasi-isomorphic to $A$: this will work).
Now, the problem is: the derived category $\mathscr{D}(\mathscr{A})$, where $\mathscr{A}$ is abelian, is in general not abelian itself. Let us consider the "queen" of abelian categories (thanks to Mitchell's embedding theorem), i.e. **Ab**= {abelian groups}. Proposition 1.1.4. tells us that if $\mathscr{D}(\mathbf{Ab})$ were triangulated and abelian, then it would be semisimple. We will later prove that the derived category of an abelian category has a natural triangulated structure, so it suffices to show that it is not semisimple in order to prove that it is not abelian. First notice that there is a canonical embedding $\mathbf{Ab} \hookrightarrow \mathscr{D}(\mathbf{Ab})$ which is fully faithful, so it suffices to find a short exact sequence which doesn't split in **Ab** to prove that $\mathscr{D}(\mathbf{Ab})$ is not semisimple. But there's plenty of them: think, for example, to $\mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \longrightarrow \mathbb{Z}_2$. Therefore, we have found a convincing example of the fact that the derived category of an abelian category is in general not abelian.
We will now give a natural triangulated structure on $\mathscr{D}(\mathscr{A})$ and later, using Proposition 1.1.4, we will be able to show that it is not abelian. To give a triangulated structure, we need first of all to give an additive equivalence $[1] : \mathscr{D}(\mathscr{A}) \longrightarrow \mathscr{D}(\mathscr{A})$. Let us give a definition:

**Definition 1.2.7.** Let $A^\bullet \in \mathscr{D}(\mathscr{A})$ be a complex. Then we define:

$$(A^\bullet[k])^i := A^{i+k} \ ; \ d^i_{A[k]} := (-1)^k d^{i+k}_A$$

and, if $f^\bullet : A^\bullet \longrightarrow B^\bullet$ is a morphism of complexes, we set

$$f[k] : A^\bullet \longrightarrow B^\bullet \ , \ f[k]^i := f^{i+k}.$$

It is easy to verify that the one defined above is actually an equivalence: an explicit inverse is given by $[-1] : \mathscr{D}(\mathscr{A}) \longrightarrow \mathscr{D}(\mathscr{A})$ , $(A^\bullet[-1])^i = A^{i-1}$, so we can generalize the definition above to $k \in \mathbb{Z}$.

**Definition 1.2.8.** Let $f^\bullet : A^\bullet \longrightarrow B^\bullet$ be a morphism in the derived category. Then the *mapping cone* of $f^\bullet$ is the complex $C(f^\bullet)$, where $C(f^\bullet)^i = A^{i+1} \oplus B^i$ and $d^i_{C(f^\bullet)} = \begin{pmatrix} -d_A^{i+1} & f^{i+1} \\ 0 & d_B^i \end{pmatrix}$.

An easy computation shows that $d_{C(f^\bullet)}$, defined as above, is actually a complex differential. There is a triangle:

$$A^\bullet \xrightarrow{f^\bullet} B^\bullet \xrightarrow{\tau} C(f^\bullet) \xrightarrow{\pi} A^\bullet[1]$$

where the two maps $\pi$ and $\tau$ are the obvious ones: $\tau^i : B^i \longrightarrow C(f^\bullet)^i = A^{i+1} \oplus B^i$ is the canonical immersion, while $\pi : C(f^\bullet)^i = A^{i+1} \oplus B^i \longrightarrow (A^\bullet)^i = A^{i+1}$ is the projection on the first factor. We can now define the class of distinct triangles:

**Definition 1.2.9.** A *distinct triangle in* $\mathscr{D}(\mathscr{A})$ is a triangle which is isomorphic to a triangle of the form $A^\bullet \xrightarrow{f^\bullet} B^\bullet \xrightarrow{\tau} C(f^\bullet) \xrightarrow{\pi} A^\bullet[1]$.

We should check that this one actually defines a class of distinct triangle, i.e. that the four axioms of the triangulated structure are verified; this would lead to the result that the natural shift functor together with the class of DT defined above is a triangulated structure over $\mathscr{D}(\mathscr{A})$. Nevertheless, the needed check-work is long and not so useful to understand what will come later, so we refer to literature (for example, Schapira-Kashiwara give a very detailed proof of it).

# Chapter 2

# Stability on triangulated and abelian categories

## 2.1 Stability functions on abelian categories

Let us recall a definition:

**Definition 2.1.1.** Let $\mathscr{A}$ be an essentially small abelian category. The *Grothendieck group* $K(\mathscr{A})$ of $\mathscr{A}$ is the abelian group generated by isomorphism classes of objects of $\mathscr{A}$, with relations of the form $[B] = [A] + [C]$ whenever $0 \longrightarrow A \longrightarrow B \longrightarrow C$ is a short exact sequence in $\mathscr{A}$.[1] Its *positive cone* $K_{>0}(\mathscr{A})$ consists of all the isomorphism classes of those elements of $\mathscr{A}$ which are not isomorphic to the zero object.

We are now ready to give the central definition:

**Definition 2.1.2.** A *stability function* $Z$ on an abelian category is a group homomorphism:

$$\begin{array}{rcl} Z \ : \ K(\mathscr{A}) & \longrightarrow & \mathbb{C} \\ \cup & & \cup \\ K_{>0}(\mathscr{A}) & \longrightarrow & \mathbb{H} \end{array}$$

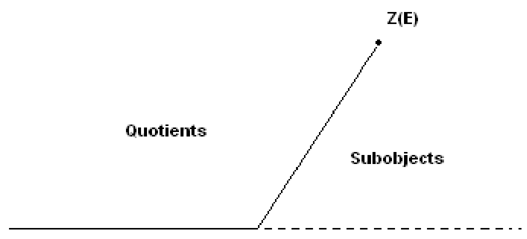
so that the positive cone $K_{>0}(\mathscr{A})$ is mapped into the upper half complex plane $\mathbb{H} = \{re^{i\pi}\phi \mid r \in (0, +\infty), \phi \in (0, 1]\}$. The image of the isomorphism class of any nonzero object $0 \neq E \in \mathscr{A}$ can therefore be written as $Z(E) = m(E)e^{i\pi\phi(E)}$, where $m(E)$ is called *mass* of $E$, while $\phi(E) = \frac{1}{\pi}\text{arg}Z(E)$ is called *phase* of $E$.

Given a stability function on $\mathscr{A}$, we are ready to give the notion of semistable and stable objects.

**Definition 2.1.3.** Let $Z : K(\mathscr{A}) \longrightarrow \mathbb{C}$ be a stability function on $\mathscr{A}$. A nonzero object $0 \neq E \in \mathscr{A}$ is called $Z$ - *semistable* (or simply *semistable*, dropping the stability function when there is no confusion) if for any $F \subset E$ subobject of $E$, $\phi(F) \leq \phi(E)$, or equivalently for every quotient $E \to G$, $\phi(E) \leq \phi(G)$. An object $E$ is called *Z-stable* (or simply *stable*) when both the inequalties in the definition are strict.

[1] The Grothendieck group $K(\mathscr{A})$ of an abelian category $\mathscr{A}$ has nothing to do with its homotopy category , even if we call them in the same way. From now on, unless differently specified, $K(\mathscr{A})$ will stand for the Grothendieck group of $\mathscr{A}$.

We can display this request as follows:




A first, very important property of semistable object, which we will try to reproduce when we will be dealing with triangulated categories, is the following one:

**Proposition 2.1.4.** *Let $Z$ be a stability function on $\mathscr{A}$, and let $E, F$ be semistable with respect to $Z$. If* $\mathrm{Hom}_{\mathscr{A}}(E, F) \neq 0$, *then* $\phi(E) \leq \phi(F)$.

*Proof.* Suppose there exists a nonzero map $f : E \longrightarrow F$. Then there are two obvious short exact sequences:

$$0 \longrightarrow \mathrm{Ker}f \longrightarrow E \rightarrow \mathrm{Coim}f \longrightarrow 0 \ ,$$

$$0 \longrightarrow \mathrm{Im}f \longrightarrow F \longrightarrow \mathrm{Coker}f \longrightarrow 0 \ .$$

Now, $E$ is semistable, $\mathrm{Ker}f \in E$ is a subobject and $E \longrightarrow \mathrm{Coim}f$ is a quotient, so it must be $\phi(\mathrm{Ker}f) \leq \phi(E)$; analogously $\phi(\mathrm{Im}f) \leq \phi(F) \leq \phi(\mathrm{Coker}f)$. But $\mathscr{A}$ is an abelian category, so $\mathrm{Im}f \cong \mathrm{Coim}f$. It follows that $\phi(E) \leq \phi(\mathrm{Coim}f) = \phi(\mathrm{Im}f) \leq \phi(F)$. □

Let us now introduce a filtration which will allow us to decompose in some sense non-semistable objects into semistable factors.

**Definition 2.1.5.** Let $E \in \mathscr{A}$ be a nonzero object, and let $Z$ be a stability condition on $\mathscr{A}$. The *Harder-Narasimhan filtration* of $E$ (shortly, H-N) with respect to Z is:

$$0 = E_0 \subset E_1 \subset ... \subset E_{n-1} \subset E_n = E$$

such that for each $j = 0, ..., n$:

1. the quotient $F_j := E_j \diagup E_{j-1}$ is $Z$-semistable;

2. The phases $\phi(F_1) > ... > \phi(F_n)$ decrease.

A stability condition $Z$ is said to have the H-N property if each nonzero object possesses the H-N filtration with respect to $Z$.

**Proposition 2.1.6.** *Let $E \in \mathscr{D}$ be nonzero. Then, the Harder-Narasimhan filtration of $E$, if exists, is unique.*

*Proof.* Assume that

$$E_\bullet = \{0 = E_0 \subset E_1 \subset ... \subset E_{n-1} \subset E_n = E\}$$

with quotients $F_i$, and

$$E'_\bullet = \{0 = E'_0 \subset E'_1 \subset ... \subset E'_{m-1} \subset E'_m = E\}$$

with quotients $F'_i$ are two Harder-Narasimhan filtrations of $E$. We can assume, without any loss of generality, that $\phi(E'_1) \geq \phi(E_1)$. Let $j$ be minimal such that $E'_1 \subset E_j$. Then the map

$$E'_1 \hookrightarrow E_j \longrightarrow E_j \diagup E_{j-1} = F_j$$

is non trivial, because $E'_1 \not\subset E_{j-1}$. Then, by the previous proposition, $\phi(E'_1) \leq \phi(F_j)$, because they are both semistable (notice that $E'_1 \cong F'_1$). But we assumed that $\phi(E_1) \leq \phi(E'_1)$, so $\phi(E_1) = \phi(F_1) \leq \phi(F_j)$, which is an absurd because the phases of semistable factors in the Harder-Narasimhan filtration strictly decrease. The only possibility is that $F_j = F_1$, i.e. $j = 1$. Therefore $E'_1 \subset E_1$, so it must be $\phi(E'_1) \leq \phi(E_1)$. This means that we can repeat the same argument by simply switching $E_\bullet$ and $E'_\bullet$, and we will find $E'_1 \cong E_1$. We can repeat the same agument for the HN filtration of $E \diagup E_1$, i.e. consider the quotient of $E_\bullet$ and $E'_\bullet$ by $E_1$. We will thus find that $F_2 := E_2 \diagup E_1 \cong E'_2 \diagup E_1 =: F'_2$. Iterating, we will get that $n = m$ and $F_i \cong F'_i$ for each $i = 1, ..., n$. This implies that $E_\bullet = E'_\bullet$. □

The following result gives a sufficient condition for the existence of the HN-filtration.

**Proposition 2.1.7.** *Let $\mathscr{A}$ be an abelian category, and let $Z$ be a stability condition on $\mathscr{A}$. Suppose that:*

1. *Neither infinite chains of subobject $... \subset E_j \subset E_{j+1} \subset ...$ such that $\phi(E_j) > \phi(E_{j+1})$ for each $j$,*
2. *Nor infinite chains of quotients $... \twoheadrightarrow E_j \twoheadrightarrow E_{j+1} \twoheadrightarrow ...$ such that $\phi(E_j) > \phi(E_{j+1})$ for each $j$*

*exist in $\mathscr{A}$. Then $Z$ has the H-N property.*

Before proving the proposition, notice that condition 1) and condition 2) imply that if $E \in \mathscr{A}$ is a nonzero object, then either $E$ is semistable or there exist a semistable subobject $A \subset E$ such that $\phi(A) > \phi(E)$, and a semistable quotient $E \longrightarrow B$ such that $\phi(E) > \phi(B)$. In fact, suppose $E$ is not semistable: then there is a nonzero subobject $F \subset E$ such that $\phi(F) > \phi(E)$. Now, if $F$ is semistable we are done, otherwise $F$ has itself a nonzero subobject $F' \subset F$ such that $\phi(F') > \phi(F) > \phi(E)$. Iterating, we find a chain of subobjects with decreasing phases. By condition 2) this chain cannot be infinite, so after a finite number of steps we find a semistable subobject $A \subset E$ with $\phi(A) > \phi(E)$. A similar argument shows the claim about quotients.
We now give a definition which will be relevant in our proof:

**Definition 2.1.8.** Let $E \in \mathscr{A}$ be nonzero. A *maximally destabilizing quotient* for $E$ (shortly, mdq) is a quotient $E \longrightarrow B$ such that if $E \longrightarrow B'$ is another quotient for $E$, then

- $\phi(B') \geq \phi(B)$;
- $\phi(B') = \phi(B)$ iff $E \longrightarrow B'$ factors through $E \longrightarrow B$:

$$\begin{array}{ccc} E & \longrightarrow & B \\ \downarrow & \circlearrowleft & \swarrow \\ B' & . & \end{array}$$

Roughly speaking, what we are looking for is the quotients with smaller phase and, amongst them, the minimal one. Let us now notice something about mdqs:

1. Mdq is unique up to isomorphism: if $E \longrightarrow B$ and $E \longrightarrow B'$ are both mdqs, of course $\phi(B) = \phi(B')$ and by definition $E \longrightarrow B'$ factors via $B$ and viceversa; this means that we have

$$E \xrightarrow{\alpha} B \xrightarrow{\beta} B', \qquad E \xrightarrow{\gamma} B'$$

$$E \xrightarrow{\gamma} B' \xrightarrow{\delta} B, \qquad E \xrightarrow{\alpha} B$$

   so $\beta \circ \alpha = \gamma$, $\delta \circ \gamma = \alpha$; therefore $\delta \circ \beta \circ \alpha = \alpha$, $\beta \circ \delta \circ \gamma = \gamma$ and finally $\delta \circ \beta = Id_B$, $\beta \circ \delta = Id_{B'}$ because $\gamma$ and $\alpha$ are both epimosphisms.

2. If $E \in \mathscr{A}$ is semistable, then $E \xrightarrow{Id} E$ is a mdq for $E$. This simply follows from the definition: if $E \longrightarrow B$ is a quotient for $E$, then $\phi(B) \geq \phi(E)$ because of the semistability of $E$.

3. If $E \longrightarrow B$ is a mdq for $E$, then $\phi(B) \leq \phi(E)$, and $\phi(B) = \phi(E)$ means $E$ semistable. In fact, $E \xrightarrow{Id} E$ is a quotient, so because of mdq property $\phi(B) \leq \phi(E)$, but if $E$ is not semistable, then there will esist a quotient of $E$, let's say $E \longrightarrow B'$ with a strictly smaller phase; then $\phi(B) \leq \phi(B') < \phi(E)$.

4. If $E \longrightarrow B$ is a mdq for $E$, then $B$ itself is semistable: in fact any quotient $B \longrightarrow B'$ for $B$ is also a quotient for $E$, by composition; then, $\phi(B') \geq \phi(B)$ by definition of mdq.

We are now ready to prove proposition 2.1.7 .

*Proof.* **STEP 1** We will show that any nonzero object has an mdq. First note that it is enough to check the mdq condition just on semistable quotients. In fact, if $E \longrightarrow B'$ is a quotient and $B'$ is not semistable, then for what we said above there exists a semistable quotient $B' \longrightarrow B''$ such that $\phi(B'') \leq \phi(B')$; $B''$ is, by composition, a quotient for $E$, too, so if $\phi(B) \leq \phi(B'')$, then in particular we will have $\phi(B) \leq \phi(B')$. Let us consider a nonzero object $E \in \mathscr{A}$. If $E$ is semistable, then $E$ possessed a mdq for what said above, i.e. $E \xrightarrow{Id} E$. If $E$ is not semistable, then there will exist a semistable subobject $A \subset E$ with $\phi(A) > \phi(E)$, as we already noticed. We can complete the inclusion morphism $A \hookrightarrow B$ to a short exact sequence:

$$0 \longrightarrow A \longrightarrow E \longrightarrow E' \longrightarrow 0 \ ,$$

with phases in decreasing order, i.e. $\phi(A) > \phi(E) > \phi(E')$.[2] We claim that a mdq for $E'$ is a mdq for $E$. Let us show this. Suppose that $E' \longrightarrow B$ is a mdq for $E$. Then $E \longrightarrow B$ is a quotient, and we need to check the mdq condition. Take another quotient $E \longrightarrow B'$, with $B'$ semistable (this will do because of what we said at the beginning of the proof). Suppose that $\phi(B') \leq \phi(B)$. We have a commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & E & \longrightarrow & E' & \longrightarrow & 0 \\ & & & \searrow \circlearrowleft & \downarrow & & & & \\ & & & & B' & & & & \end{array} ,$$

where the arrow $A \longrightarrow B'$ is just the composition of $A \longrightarrow E$ and $E \longrightarrow B'$. We have $\phi(E') \leq \phi(B)$ because $E' \longrightarrow B$ is a mdq, $\phi(B) \geq \phi(B')$ for what we have just supposed, and $\phi(A) > \phi(E) > \phi(E')$, therefore, reading the long chain of inequalties, we get that $\phi(A) > \phi(B')$. But $A$ and $B'$ are both semistable, then $\mathrm{Hom}_{\mathscr{A}}(A, B') = 0$. This means that the composition of $A \longrightarrow E$ and $E \longrightarrow B'$ must be zero, therefore, because of the universal property of the kernel, the arrow $E \longrightarrow B'$ factors through $E'$:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & E & \longrightarrow & E' & \longrightarrow & 0 \\ & & & \searrow \circlearrowleft & \downarrow & \swarrow & & & \\ & & & & B' & & & & \end{array} .$$

Thus, $E' \longrightarrow B'$ is also a quotient for $E'$: then, by definition of mdq, we have that $\phi(B) \leq \phi(B')$. But we supposed that $\phi(B) \leq \phi(B')$, then $\phi(B) = \phi(B')$. So, $E \longrightarrow B$ is a mdq. We can now replace $E$ by $E'$ and repeat the argument: if $E'$ is semistable, the proof ends, otherwise there is another short exact sequence like the one we built up for $E$. Iterating, we find a chain of quotients with descending phases: this process must end by condition 2), then after a finite number of steps we will find a semistable object B:

$$E \longrightarrow E' \longrightarrow ... \longrightarrow E^{(n)} \longrightarrow B$$

which will be a mdq for $E^{(n)}$. But, for what we proved above, a mdq for $E^{(n)}$ is also a mdq for $E^{(n-1)}$ and therefore, iterating, a mdq for $E$.

**STEP 2** We will now build explicitely the H-N filtration. Let $E$ be a nonzero object. If $E$ is semistable, the H-N filtration of $E$ is $0 \subset E$. Else, because of Step 1, E possesses a mdq $E \longrightarrow B$ with $\phi(B) < \phi(E)$. We complete the quotient map to a short exact sequence:

[2] It is obvious that $\phi(E) > \phi(E')$: within the Grothendieck group $K(\mathscr{A})$ any short exact sequence splits, so $[E] = [A] + [E']$. Moreover, $Z$ is an additive group homomorphism, so we will have $Z(E) = Z(A) + Z(E')$: this means that the phase of $E$ ranges between the phases of $A$ and $E'$ respectively.

$$0 \longrightarrow E' \longrightarrow E \longrightarrow B \longrightarrow 0$$

with phases $\phi(E') > \phi(E) > \phi(B)$. Now, let us suppose that $E' \longrightarrow B'$ is a mdq. We can complete this quotient map, too, to a short exact sequence $0 \longrightarrow K \longrightarrow E' \longrightarrow B' \longrightarrow 0$; therefore we have a diagram:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K & \longrightarrow & E' & \longrightarrow & B' & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K & \longrightarrow & E & \longrightarrow & Q & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & B & = & B & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

where $E \longrightarrow Q$ completes the inclusion map $K \longrightarrow E$ ($K$ is a subobject of $E'$ which is a subobject of $E$ itself) to a short exact sequence. All the rows and the first two columns are exact and, using some homological algebra, it is easy to show that the third column is exact, as well. Now, $E \longrightarrow Q$ is a quotient of $E$, so $\phi(Q) \geq \phi(B)$ because $E \longrightarrow B$ is a mdq, and it cannot be $\phi(Q) = \phi(B)$ because this would mean that $E \longrightarrow Q$ factors via $B$, i.e. $E \longrightarrow B \longrightarrow Q$. But the diagram tells us that $E \longrightarrow B$ factors via $Q$, so we would have $Q \cong B$ and $B' \cong 0$, which is absurd because we assumed $B'$ to be a mdq for $E'$. Therefore $\phi(Q) > \phi(B)$ and, by the exactness of the last column, $\phi(B') > \phi(B)$. We have a chain of three subobjects:

$$K \subset E' \subset E$$

with semistable quotients $E'/K = B'$ and $E/E' = B$ such that $\phi(B') > \phi(B)$. We can now replace $E$ by $E'$ and $E'$ by $K$ and iterate until we find a semistable object, i.e. in the diagram above this happens when $B' \cong E'$ and $K = 0$ (we are sure to find such a situation after a finite number of steps because of condition 1) ). Therefore, we can rename objects simply setting $E' = E_{n-1},\ K = E_{n-2},\ B = F_n,\ B' = F_{n-1}$ and so on. At the end, we will have:

$$0 \subset E_0 \subset E_1 \subset ... \subset E_{n-1} \subset E_n = E$$

with $E_j/E_{j-1} = F_j$ and $\phi(F_j) > \phi(F_{j+1})$ for all $j$. This is precisely the H-N filtration of $E$.

□

## 2.2 t-structures on triangulated categories

In the first section we have seen that the concept of stability function on an abelian category arises in a quite natural way, and we would now like to weaken this construction in order to make it adapt to an arbitrary triangulated category. A few problems arise immediatly, though: we cannot give a straightforward definition of semistable object, which strongly involves the ideas of subobject and quotient in abelian categories. We need therefore to replace the partial ordering on objects, given by inclusion, with something else. We will see that a key role in the machinery which will allow us to solve this problem is played by t-structures, which we are now going to introduce. Let us start with a definition:

**Definition 2.2.1.** Let $\mathscr{D}$ be a triangulated category. A *t-structure* on $\mathscr{D}$ is given by a full additive subcategory $\mathscr{F} \subset \mathscr{D}$ which verifies the following conditions:

1. $\mathscr{F}[1] \subset \mathscr{F}$ (i.e., $\mathscr{F}$ is closed under positive shifts);
2. for each nonzero object $E \in \mathscr{D}$ there exists a triangle $F \longrightarrow E \longrightarrow G$ where $F \in \mathscr{F}$ , $G \in \mathscr{F}^{\perp}$ and $\mathscr{F}^{\perp} := \{G \in \mathscr{D} \mid \mathrm{Hom}_{\mathscr{D}}(F,G) = 0\ \forall F \in \mathscr{F}\}$. (i.e., the category $\mathscr{F}$ decomposes nonzero objects).

Let us give a basic example:

**Example 2.2.2.** Let $\mathscr{D} = \mathscr{D}(\mathscr{A})$ be the derived category of an abelian category. There exists on $\mathscr{D}$ a *trivial t-structure*, which is given by

$$\mathscr{F} := \{A^\bullet \in \mathscr{D}(\mathscr{A}) \mid H^i(A^\bullet) = 0\ \forall i > 0\}\ ,$$

i.e., $\mathscr{F}$ consists of the complexes whose cohomology is concentrated in negative degree. The category $\mathscr{F}$ is obviously closed under positive shift (recall that what we actually do is moving backward the whole complex, so if the cohomology of $A^\bullet$ is zero from degree one on, then the cohomology of $A^\bullet[1]$ will be zero from degree zero on), and its orthogonal is given by:

$$\mathscr{F}^{\perp} := \{A^\bullet \in \mathscr{D}(\mathscr{A}) \mid H^i(A^\bullet) = 0\ \forall i < 1\},$$

i.e., the complexes whose cohomology is concentrated in strictly positive degree.

**Definition 2.2.3.** Let $\mathscr{F}$ be a t-structure on $\mathscr{D}$. The *heart*[3] of $\mathscr{F}$ is the subcategory

$$\mathscr{H} := \mathscr{F} \cap \mathscr{F}^{\perp}[1]$$

**Example 2.2.4.** Consider the trivial t-structure on the derived category $\mathscr{D}(\mathscr{A})$ of an abelian category $\mathscr{A}$ (Example 2.2.2). Then, its heart is:

$$\mathscr{H} = \mathscr{F} \cap \mathscr{F}^{\perp}[1] = \{A^\bullet \in \mathscr{D}(\mathscr{A}) \mid H^i(A^\bullet) = 0\ \forall i > 0\} \cap \{A^\bullet \in \mathscr{D}(\mathscr{A}) \mid H^i(A^\bullet) = 0\ \forall i < 0\} =$$

[3] The category $\mathscr{H}$ is defined as the intersection of two subcategories. Now, it is a very controversial point to define what intersection means when we are dealing with classes instead of sets (we didn't assume our category to be essentially small), and it ought to be clarified. This one is an example of how unstable and *in fieri* the formalization of the whole theory is.

$$= \{A^\bullet \in \mathscr{D}(\mathscr{A}) \mid H^i(A^\bullet) = 0\ \forall i \neq 0\} \cong \mathscr{A}$$

which gives us a canonical way to embed an abelian category in its derived category: as the heart of the trivial t-structure.

**Definition 2.2.5.** A t-structure $\mathscr{F}$ on $\mathscr{D}$ is *bounded* if

$$\mathscr{D} = \bigcup_{i,j\in\mathbb{Z}} \mathscr{F}[i] \cap \mathscr{F}^{\perp}[j]$$

or, equivalently, if $\mathscr{F}$ is the closed-extension[4] category generated by $\{\mathscr{H}[j]\}_{j\in\mathbb{Z}}$.

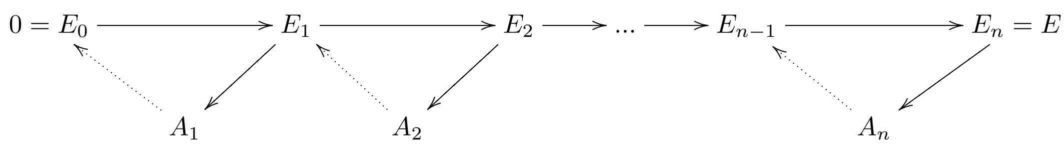
**Example 2.2.6.** Note that, in the case of Example 2.2.2., the trivial t-structure is *not* bounded. In fact, this would be equivalent to ask any complex to have nonzero cohomology just inside a bounded interval of integer values. As we already know, this condition is not always satisfied. Nonetheless, if we consider the trivial t-structure restricted to the *bounded* derived category, what we obtain is actually a bounded t-structure.

Let us now prove a result which will show us a nice way to reproduce what we have done for abelian categories.

**Proposition 2.2.7.** *Let $\mathscr{D}$ be a triangulated category, and let $\mathscr{H}$ be an additive full subcategory. Then $\mathscr{H}$ is the heart of a bounded t-structure on $\mathscr{D}$ if and only if the following conditions hold:*

1. *If we consider any two integers $k_1 > k_2 \in \mathbb{Z}$, then $\mathrm{Hom}_{\mathscr{D}}(A_1[k_1], A_2[k_2]) = 0$ for all $A_1, A_2 \in \mathscr{H}$;*

2. *For each nonzero object $E \in \mathscr{D}$ there exist a finite sequence of decreasing integers $k_1 > ... > k_n$ and a collection of distinct triangles:*



*where $A_j \in \mathscr{H}[k_j]$ for all $j = 1, ..., n$.*

Before proving the proposition, we will need a few lemmas:
**Lemma 1** *If $\mathscr{C} \subset \mathscr{D}$ is an extension-closed subcategory, then all its shifts are extension-closed.*

*Proof.* Consider the following DT:

$$A \longrightarrow E \longrightarrow B \xrightarrow{+1}$$

with $A, B \in \mathscr{C}[i]$, $E \in \mathscr{D}$. Then, using the fact that the shift functor and its inverse are equivalences, we get that the triangle $A[-i] \longrightarrow E[-i] \longrightarrow B[-i] \xrightarrow{+1}$ is distinct, too. Now, $A[-i]$ and $B[-i]$ are in $\mathscr{C}$, therefore $E[-i]$ is in $\mathscr{C}$, too, because $\mathscr{C}$ is extension-closed. But then $E$ is in $\mathscr{C}[i]$. □

[4] We say that a category $\mathscr{C}$ is the extension-closed category generated by a collection of categories $\mathscr{A}_j$ if $\bigcup \mathscr{A}_j \subset \mathscr{C}$ and $\mathscr{C}$ is closed with respect to distinct triangles, i.e. if whenever any two vertices of a distinct triangle are in $\mathscr{C}$, so is the third.

**Lemma 2** *If $\mathscr{C} \subset \mathscr{D}$ is an extension-closed subcategory, then its orthogonal is extension-closed, as well.*

*Proof.* Take a DT:

$$A \longrightarrow E \longrightarrow B \xrightarrow{+1}$$

with $A, B \in \mathscr{C}^{\perp}$, $E \in \mathscr{D}$, and let $X$ be a nonzero object in $\mathscr{C}$. By property 2, there is a long exact sequence:

$$... \longrightarrow \mathrm{Hom}_{\mathscr{D}}(X, A) \longrightarrow \mathrm{Hom}_{\mathscr{D}}(X, E) \longrightarrow \mathrm{Hom}_{\mathscr{D}}(X, B) \longrightarrow ... \,.$$

Now, $\mathrm{Hom}_{\mathscr{D}}(X, A) = \mathrm{Hom}_{\mathscr{D}}(X, B) = 0$ by orthogonality, therefore $\mathrm{Hom}_{\mathscr{D}}(X, E) = 0$, too. By the arbitrary choice of $X \in \mathscr{C}$, we can conclude that $E \in \mathscr{C}^{\perp}$.

□

*Proof.* First suppose that $\mathscr{H}$ is the heart of a t-structure $\mathscr{F}$ on $\mathscr{D}$.

1. The shift functor, its inverse and their powers are all equivalences; in particular, they are fully faithful. Therefore, if $A$ and $B$ are in $\mathscr{H}$:

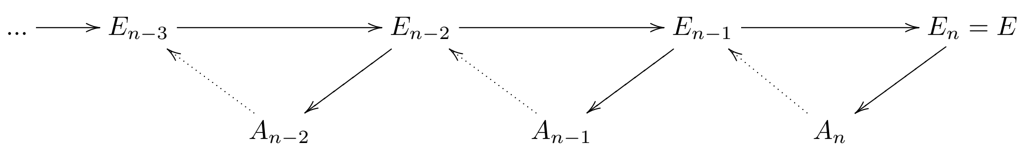
$$\mathrm{Hom}_{\mathscr{D}}(A[k_1], B[k_2]) \cong \mathrm{Hom}_{\mathscr{D}}(A[k_1 - k_2 - 1], B[-1]).$$

   Now, $k_1 > k_2$, so $k_1 - k_2 - 1 \geq 0$: it means that $A[k_1 - k_2 - 1]$ is in $\mathscr{F}[n]$ for some $n \geq 0$. But $\mathscr{F}$ is closed under shift, therefore $A[k_1 - k_2 - 1]$ is in $\mathscr{F}$. On the other hand, $B$ is in $\mathscr{H} = \mathscr{F} \cap \mathscr{F}^{\perp}[1]$, so $B[-1]$ is in $(\mathscr{F} \cap \mathscr{F}^{\perp}[1])[-1] = \mathscr{F}[-1] \cap \mathscr{F}^{\perp} \subset \mathscr{F}^{\perp}$. Therefore $B$ is in $\mathscr{F}^{\perp}$ and, by definition of t-structure, $\mathrm{Hom}_{\mathscr{D}}(A[k_1 - k_2 - 1], B[-1]) = 0$.

2. Consider a nonzero object $E \in \mathscr{D}$. We want to build up the HN filtration of $E$. First of all, notice that the boundedness of $\mathscr{F}$ implies that $E \in \mathscr{F}[i] \cap \mathscr{F}^{\perp}[j]$ for some $i, j \in \mathbb{Z}$, then take $E[-i-1] \in \mathscr{D}$. By definition of t-structure, there exists a DT $A \longrightarrow E[-i-1] \longrightarrow B \xrightarrow{+1}$ with $A \in \mathscr{F}, B \in \mathscr{F}^{\perp}$. By the fact that the shift functor is an equivalence, we get that the triangle $A[i+1] \longrightarrow E \longrightarrow B[i+1] \xrightarrow{+1}$ is distinguished. Now, $E \in \mathscr{F}[i]$ by hypotesis, $A[i+2] \in \mathscr{F}[i+2] \subset \mathscr{F}[i]$, the triangle $E \longrightarrow B[i+1] \longrightarrow A[i+2]$ is distinguished by the first axiom, therefore, using the fact that $\mathscr{F}$ is extension-closed (and so are all its shift by Lemma 1), we can conclude that $B[i+1] \in \mathscr{F}[i] \cap \mathscr{F}^{\perp}[i+1] = \mathscr{H}[i]$. We have so found a map $E \longrightarrow B_i$, where $B_i := B[i+1] \in \mathscr{H}[i]$ and, renaming objects in a suitable way, we have a DT:

$$E_{n-1} \longrightarrow E_n \longrightarrow A_n \,.$$

   We can now replace $E_n$ with $E_{n-1}$ and iterate. Eventually, we find a filtration:



where $A_j \in \mathscr{H}[k_j]$ for all $j = 1, ..., n$.

What we need to check is:

(a) that the filtration is actually bounded, i.e. that after a finite number of steps we actually find zero;

(b) that $k_1 > ... > k_n$.

Consider the DT we had at the beginning:

$$A' \longrightarrow E \longrightarrow B' \xrightarrow{+1}$$

(of course $A' = A[i+1]$, $B' := B[i+1]$) where we supposed $E$ to be in $\mathscr{F}[i] \cap \mathscr{F}^{\perp}[j]$. Notice that if $E$ is nonzero the condition $i \leq j-1$ must hold: indeed, suppose that $A_s \in \mathscr{H}[k_s]$, $s = 1, ..., n$ are the semistable quotients in the HN filtration of E, then we know that:

$$\mathscr{F}[i] = \{E \in \mathscr{D} \mid k_s \geq i \ \forall s = 1, ..., n\}$$

$$\mathscr{F}^{\perp}[j] = \{E \in \mathscr{D} \mid k_s \leq j-1 \ \forall s = 1, ..., n\}$$

therefore, for $E$ to be in $\mathscr{F}[i] \cap \mathscr{F}^{\perp}[j]$, it must be $i \leq k_s \leq j-1$. We want to show that $A' \in \mathscr{F}[i'] \cap \mathscr{F}^{\perp}[j']$ with $j' \leq j$, $i' > i$. In fact, we had found that $B' \in \mathscr{F}^{\perp}[i+1]$ (so $B[-1] \in \mathscr{F}^{\perp}[i]$), $E \in \mathscr{F}^{\perp}[j]$, so recalling that $i \leq j-1$, i.e. $i < j$ we get a DT, $B'[-1] \longrightarrow A' \longrightarrow E \xrightarrow{+1}$ where both $B'[-1]$ and $A'$ are in $\mathscr{F}^{\perp}[j]$ by Lemma 3. Moreover, $A' = A[i+1]$, so $A' \in \mathscr{F}[i+1]$, so $i' = i+1 > i$. Therefore the quotient of $A'$ is in $\mathscr{H}[i']$ with $i' > i$, and after a finite number of steps we find zero because at a certain point the object completing the triangle will be in $\mathscr{F}[i^{(n)}] \cap \mathscr{F}^{\perp}[j^{(n)}]$ with $i^{(n)} \geq j^{(n)} - 1$.

For the converse, suppose there exists a full additive subcategory which verifies conditions 1) and 2). Define $\mathscr{F}$ as the extension closed subcategory generated by $\{\mathscr{H}[j], j \in \mathbb{Z}_{\geq 0}\}$. Moreover, the category $\mathscr{F}^{\perp}$ will be the extension closed subcategory generated by $\{\mathscr{H}[j], j \in \mathbb{Z}_{<0}\}$: in fact, the quotients in the HN filtration of an object $F \in \mathscr{F}$ will all be in negative shifts of $\mathscr{H}$, while the quotients in the HN filtration of an object $G \in \mathscr{F}^{\perp}$ will all be in positive shifts of $\mathscr{H}$; therefore there can be no nonzero map between $F$ and $G$ because if it existed, it would exist a nonzero map from the quotients of $F$ to the quotients of $G$, as well, and we know it can't be by condition 1). This proves that $ECS\langle \mathscr{H}[j], j \in \mathbb{Z}_{<0} \rangle \subset \mathscr{F}^{\perp}$. For the reverse inclusion, consider an object $E \notin ECS\langle \mathscr{H}[j], j \in \mathbb{Z}_{<0} \rangle$. Then, if the HN filtration is as in condition 2), $k_1$ must be positive. This means that $A_1 \in \mathscr{F}$ and there is a nonzero map between $A_1$ and $E$ (the composition of the horizontal arrows). Therefore, $E$ cannot be in $\mathscr{F}^{\perp}$. Let us now prove that $\mathscr{F}$, as defined above, is a t-structure on $\mathscr{D}$ (the boundedness comes straight from the definition). It is obvious that $\mathscr{F}$ is closed under positive shift: in fact if $E$ is in $\mathscr{F}[1]$, then $E[-1] \in \mathscr{F}$, so it has a HN filtration whose semistable quotients are in positive shifts of the heart . Let

$$E_{n-1} \longrightarrow E[-1] \longrightarrow A_n \xrightarrow{+1}$$

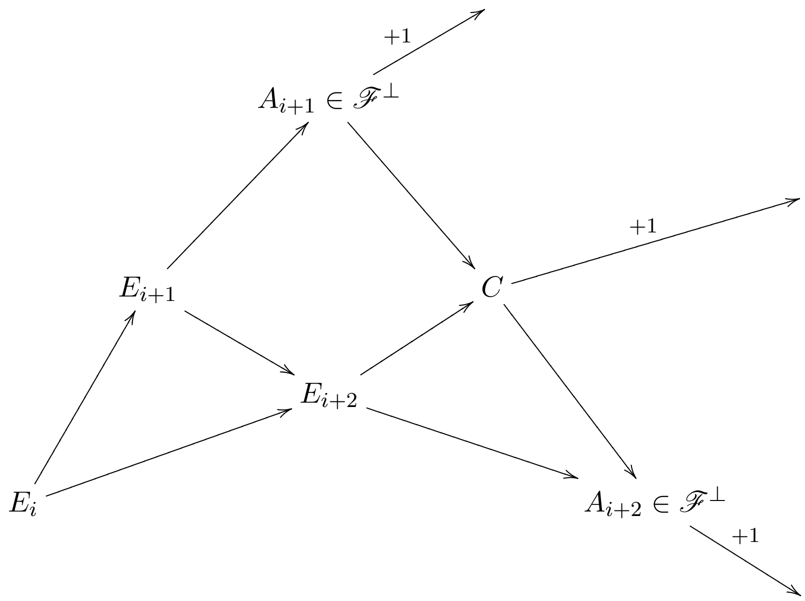
be the last triangle of this filtration, with $A_n \in \mathscr{H}[k_n]$, for some $k_n \geq 0$. Then, by shifting, we get $E_{n-1}[1] \longrightarrow E \longrightarrow A_n[1] \xrightarrow{+1}$. Now, $A_n[1] \in \mathscr{H}[k_n{+}1] \subset \mathscr{F}$, because $k_n + 1$ is obviously positive. So, by the fact thet $\mathscr{F}$ is extension-closed, it suffices to prove that $E_{n-1}[1] \in \mathscr{F}$ to show that $E$ is in $\mathscr{F}$, too. Consider the last triangle but one:

$$E_{n-2} \longrightarrow E_{n-1} \longrightarrow A_{n-1} \xrightarrow{+1}$$

where $A_{n-1} \in \mathscr{H}[k_{n-1}],\ k_{n-1} > 0$. Again, by shifting we find that $E_{n-2}[1] \longrightarrow E_{n-1}[1] \longrightarrow A_{n-1}[1] \xrightarrow{+1}$, so $A_{n-1}[1] \in \mathscr{H}[k_{n-1}] \subset \mathscr{F}$ and it suffices to prove that $E_{n-2}[1] \in \mathscr{F}$ to show that $E_{n-1}[1]$, and therefore $E$, as well, are in $\mathscr{F}$. Going backwards this way we reach the first triangle:

$$E_1 \cong A_1 \longrightarrow E_2 \longrightarrow A_2 \xrightarrow{+1},$$

where $A_1 \in \mathscr{H}[k_1],\ A_2 \in \mathscr{H}[k_2]$ with $k_1, k_2 > 0$ By shifting, we get $A_1[1] \longrightarrow E_2[1] \longrightarrow A_2[1] \xrightarrow{+1}$. But $A_1[1] \in \mathscr{H}[k_1 + 1]$ and $A_2 \in \mathscr{H}[k_2 + 1]$ are both in $\mathscr{F}$, so $E_2[1]$ is in $\mathscr{F}$, as well. But then, considering the triangle $E_2[1] \longrightarrow E_3[1] \longrightarrow A_3[1] \xrightarrow{+1}$ we get that $E_3[1]$ is in $\mathscr{F}$ and, proceeding this way, $E$ is in $\mathscr{F}$, too. Therefore $\mathscr{F}$ is closed under shift. Moreover, to build up the DT $F \longrightarrow E \longrightarrow G \xrightarrow{+1}$ for some $E \in \mathscr{D}, F \in \mathscr{F}, G \in \mathscr{F}^{\perp}$ it suffices to look at the HN filtration of $E$: if the quotients are all in positive shifts of $\mathscr{H}$, then $E$ is in $\mathscr{F}$ and the DT $E \longrightarrow E \longrightarrow 0 \xrightarrow{+1}$ is as above; if instead the quotient of $E$ are all in negative shifts of $\mathscr{H}$, then $E$ is in $\mathscr{F}^{\perp}$ and the DT $0 \longrightarrow E \longrightarrow E \xrightarrow{+1}$ meets the requirements we need; if finally some of the quotients are in negative shifts while the others are in positive shifts of $\mathscr{H}$, there is a little bit of work to do. Suppose $k_i \geq 0,\ k_{i+1} < 0$. This means that $E_i \in \mathscr{F}$. Consider the map $E_i \longrightarrow E$. We want to show that its cone, which we call $E'$, is in $\mathscr{F}^{\perp}$. Now, consider the composition $E_i \longrightarrow E_{i+1} \longrightarrow E_{i+2}$. The cone of $E_i \longrightarrow E_{i+1}$ is $A_{i+1} \in \mathscr{H}[k_{i+1}] \subset \mathscr{F}^{\perp}$ because we supposed $k_{i+1} < 0$, and the same holds for the cone of $E_{i+1} \longrightarrow E_{i+2}$, which is $A_{i+2} \in \mathscr{H}[k_{i+2}] \in \mathscr{F}^{\perp}$. Therefore, the octahedron axiom shows us that:




therefore, by the fact that $\mathscr{F}^\perp$ is extension-closed, the cone $C$ of the map $E_{i+1} \longrightarrow E_{i+2}$ is in $\mathscr{F}^\perp$. Repeat the argument with the composition $E_i \longrightarrow E_{i+2} \longrightarrow E_{i+3}$. Again, the cone of $E_i \longrightarrow E_{i+2}$ is in $\mathscr{F}^\perp$ because of what we have just shown, and the cone of $E_{i+2} \longrightarrow E_{i+3}$, which is $A_{i+3} \in \mathscr{H}[k_{i+3}]$ is in $\mathscr{F}^\perp$, too. Therefore, the cone of $E_i \longrightarrow E_{i+3}$ is the central vertex of a DT whose extremal vertices are both in $\mathscr{F}^\perp$, so it is in $\mathscr{F}^\perp$, too. We can iterate until we find:

$$E_i \longrightarrow E_{n-1} \longrightarrow E.$$

The cone of $E_i \longrightarrow E_{n-1}$ is in $\mathscr{F}^\perp$ by induction, the cone of $E_{n-1} \longrightarrow E$ is in $\mathscr{F}^\perp$, too, because it is $A_n \in \mathscr{H}[k_n]$, therefore by the octahedron axiom again the cone of $E_i \longrightarrow E$, which we called $E'$, is in $\mathscr{F}^\perp$, as well.[5]

□

Notice that this result shows a sort of triangulated version of what happened in abelian categories: if we think of the objects in the categories $\mathscr{H}[j]$ as the old semistable objects, with new phases given by the $j$'s, we just get that there are no nonzero morphisms between two semistable objects if the first one has a bigger phase, and that any nonzero object possesses a Harder-Narasimhan filtration. However, there is an important difference: here, the phases are indexed by integers, while in abelian categories they were indexed by reals. Therefore, we need to "slice" further our category. The tool which will allow us to do this is called, for obvious reasons, *slicing*.

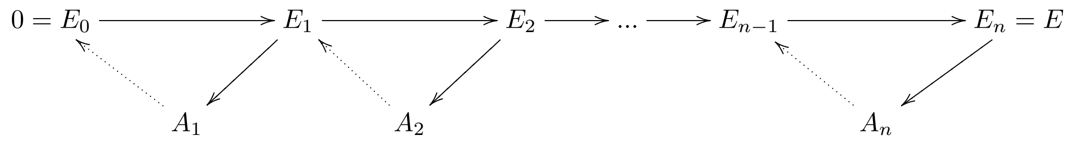
**Definition 2.2.8.** Let $\mathscr{D}$ be a triangulated category. A *slicing* on $\mathscr{D}$ is a collection of full additive subcategories $\mathscr{P}(\phi)$, $\phi \in \mathbb{R}$, such that the following conditions hold:

[5] Thanks to Donatella Iacono for helping me with this proof.

**a)** $\mathscr{P}(\phi+1) = \mathscr{P}(\phi)[1]$;

**b)** if $A \in \mathscr{P}(\phi)$, $B \in \mathscr{P}(\psi)$, for some $\phi, \psi \in \mathbb{R}$, and $\phi > \psi$, then $\mathrm{Hom}(A, B) = 0$;

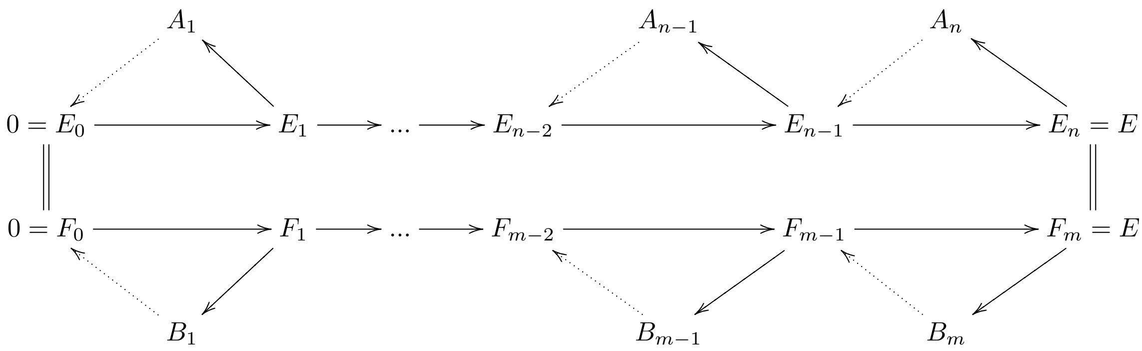
**c)** for each nonzero object $\mathbb{E} \in \mathscr{D}$ there exist a finite sequence of decreasing reals $\phi_1 > ... > \phi_n$ and a collection of distinguished triangles




where $A_i \in \mathscr{P}(\phi_i)$ $\forall i = 1, ..., n$.

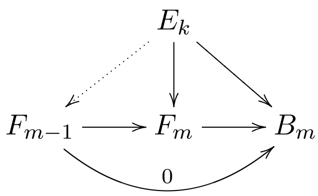
**Lemma 2.2.9.** *The Harder-Narasimhan filtration in* c) *is unique.*

*Proof.* Fix a nonzero object $E \in \mathscr{D}$. Suppose there are two such filtrations:

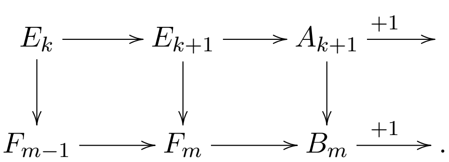



with $A_i \in \mathscr{P}(\phi_i)$ , $\phi_i > ... > \phi_n$ and $B_i \in \mathscr{P}(\psi_i)$ , $\psi_i > ... > \psi_m$.

**STEP 1** Consider the minimum $k$ for which the composition $E_k \longrightarrow E_n \longrightarrow B_m$ is zero (notice that the composition $E_{n-1} \longrightarrow E_n = F_m \longrightarrow B_m$ is zero for the first property of distinguished triangles. ). We have that the following holds:

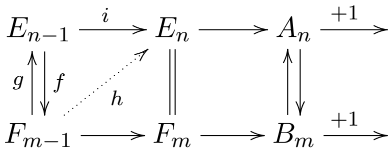



i.e., the map from $E_k$ to $F_m$ lifts to $F_{m-1}$. Therefore, we have a commutative diagram:

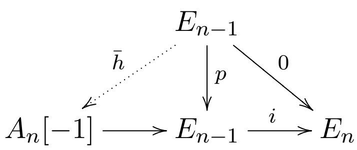


The arrow $E_{k+1} \longrightarrow F_m$ is nonzero, because we supposed $k$ to be minimum, therefore the map $A_{k+1} \longrightarrow B_m$ is nonzero, as well. By the second condition in the definition of a slicing, we get that if $A_{k+1} \in \mathscr{P}(\phi_{k+1}, B_m \in \mathscr{P}(\psi_m)$, then $\phi_n < ... < \phi_{k+1} < \psi_m$. We can now repeat the argument by switching the two filtration in order to find $\psi_m < ... < \psi_{j+1} < \phi_n$. Thus, the only possibility is that $k+1 = m = n$, i.e. $m = n,\ k = n-1$ and $\phi_n = \psi_m$.

**STEP 2** Consider the following diagram:



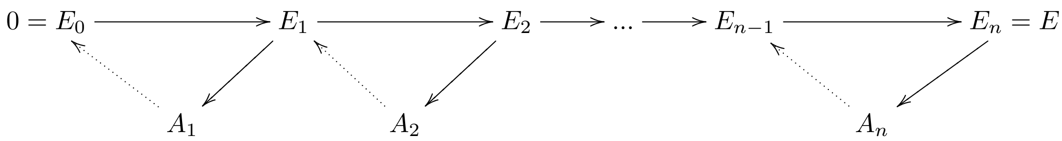

(which is simply the diagram above, where $k = n-1$ and we obtain the parallel arrows by switching the two lines). Now, $i \circ g = h$, $h \circ f = i$, therefore $h \circ f \circ g = h$ and symmetrically $i \circ g \circ f = i$. This implies that $i \circ (g \circ f - Id) = 0$ and $h \circ (f \circ g - Id) = 0$. Set $p := g \circ f - Id \ : \ E_{n-1} \longrightarrow E_{n-1}$, and consider the following:




we get that there is a map $E_{n-1} \xrightarrow{\bar{h}} A_n[-1]$. But $E_{n-1} \in \mathscr{P}(> \phi_n)$ and $A_n[-1] \in \mathscr{P}(\phi_n - 1)$, therefore $\bar{h} = 0$. But then $p = 0$ and $g \circ f = Id$. We can do the same with $h \circ (g \circ f - Id)$, and we will get $E_{n-1} \cong F_{m-1}$. The assertion then follows by simply iterating.

□

**Definition 2.2.10.** Let $E$ be a nonzero object of $\mathscr{D}$. Then, if the H-N filtration of $E$ is




we can define the *maximal* and the *minimal phase* of $E$, i.e.:

$$\phi^+_{\mathscr{P}}(E) = \phi_1$$

$$\phi^-_{\mathscr{P}}(E) = \phi_n$$

respectively. We will often write $\phi^{\pm}(E)$ omitting the slicing when there is no confusion.

Notice that maximum and minimum phase of a nonzero object are well defined because of the uniqueness of the Harder-Narasimhan filtration, and that $\phi^-_{\mathscr{P}}(E) \leq \phi^+_{\mathscr{P}}(E)$ (the equality holds if and only if $E$ is in $\mathscr{P}(\phi)$, where $\phi = \phi^+_{\mathscr{P}}(E) = \phi^+_{\mathscr{P}}(E)$).

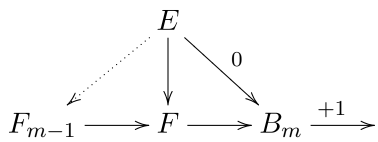
**Definition 2.2.11.** We define the categories $\mathscr{P}(I)$, where $I$ is an interval whose length is less or equal than 1, as the extension closed category generated by $\{\mathscr{P}(\phi)\}_{\phi\in I}$. In particular, we have:

$$\mathscr{P}((a,b)) = \{0 \in \mathscr{D}\} \cup \{E \in \mathscr{D} \ s.t. \ a < \phi^-_{\mathscr{P}}(E) \leq \phi^+_{\mathscr{P}}(E) < b\}$$

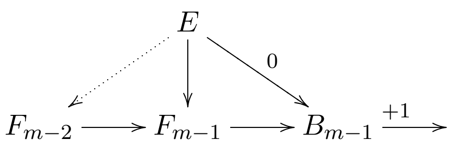
**Lemma 2.2.12.** *If* $E, F \in \mathscr{D}$ *are nonzero and* $E \in \mathscr{P}(> \phi), F \in \mathscr{P}(< \psi)$ *where* $\phi \geq \psi$*, then* $\mathrm{Hom}(A, B) = 0$*.*

*Proof.* Call $B_i, \ i = 1, ..., m$ the semistable factors in the HN filtration of $F$, $\psi_i$ their phases and consider the integer $n$ =length of the H-N filtration of E, which is finite and well defined. We make induction on $n$:

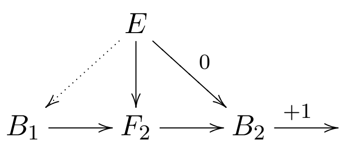
$n = 1$. If $n = 1$, then $\phi^+(E) = \phi^-(E) = \phi(E)$ and $E \in \mathscr{P}(\phi(E))$. Suppose there is a map $E \longrightarrow F$. Then, by composition, there is also a map $E \longrightarrow F \longrightarrow B_m$ (the last semistable factor). But $B_m$ is in $\mathscr{P}(\psi_m)$, and $\psi_m < \psi_1 < \psi \leq \phi(E)$, therefore, by definition of slicing, this composition must be zero. We thus have the following diagram:




which shows that the map $E \longrightarrow F$ lifts to $F_{m-1}$. Now, if the map $E \longrightarrow F_{m-1}$ is zero, the proof is finished, because it means that the map $E \longrightarrow F$ is zero, as well. if it is not zero, on the other hand, we can iterate the argument in this way:

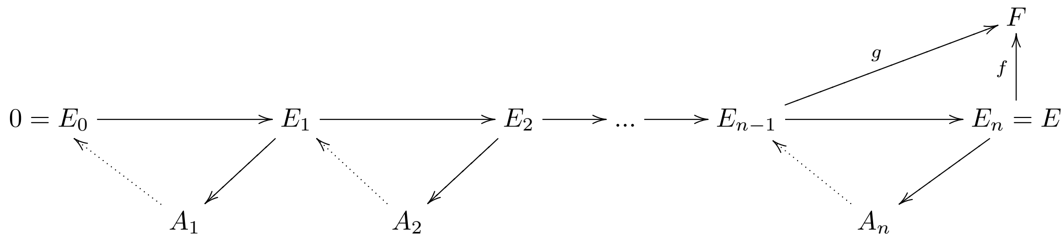



and find a map $E \longrightarrow F_{m-2}$. If this map is zero, the proof is finished, for would mean that the map $E \longrightarrow F_{m-1}$ and therefore the map $E \longrightarrow F$ were both zero. If it is not zero, we iterate again. At the end, we will find such a diagram:

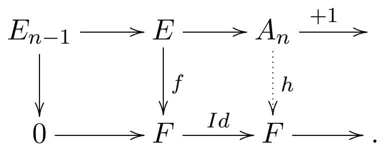



Now, $B_1$ is in $\mathscr{P}(\psi_1)$, where $\psi_1 < \psi \leq \phi(E)$. Therefore the map $E \to B_1$ is zero. But it means that $E \longrightarrow F_2$ is zero, too and, going backwards, also the map $E \longrightarrow F$.

$n > 1$. Suppose there exists $f : E \longrightarrow F$:




where $g$ is simply the composition $E_{n-1} \longrightarrow E$ and $E \xrightarrow{f} F$. By induction, $g = 0$.Therefore there is a commutative diagram:




But $A_n \in \mathscr{P}(\phi_n)$, and $\phi_n = \phi^-_{\mathscr{P}}(A) > \phi \geq \psi$, therefore $h = 0$ and $f = 0$ by commutativity.

$\square$

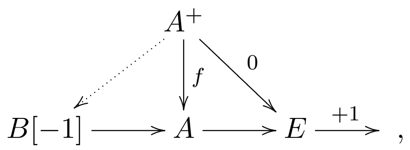
**Lemma 2.2.13.** *Let $\mathscr{P}$ be a slicing on $\mathscr{D}$ and let $I$ be an interval with lengh $\ell(I) \leq 1$. Suppose there exists a DT*

$$A \longrightarrow E \longrightarrow B \xrightarrow{+1}$$

*with $A, E, B \in \mathscr{P}(I)$. Then:*

1. $\phi^+(A) \leq \phi^+(E)$ ;
2. $\phi^-(E) \leq \phi^-(B)$

*Proof.* Suppose $I = [t, t+1]$ for some $t \in \mathbb{R}$[6] and set $\phi = \phi(A)$. By definition of slicing, there exist an object $A^+ \in \mathscr{P}(\phi)$ and a nonzero map $A^+ \longrightarrow A$ (simply look at the H-N filtration of A: $A^+ := A_1$, and the map is the composition of the horizontal arrows). If we suppose that $\phi > \phi^+(E)$, then there are no nonzero morphisms from $A^+ \longrightarrow E$, therefore the following diagram commutes for some map $A^+ \longrightarrow B[-1]$:




[6]Notice that we never use the actual length of the interval, so it is not restrictive to suppose it to be exactly one.

i.e., $f$ factors via $B[-1]$. But, by assumption, $B[-1]$ is in $\mathscr{P}(\leq t)$, so it must be $\phi \leq t$, because otherwise the map $A^+ \longrightarrow B[-1]$ would be zero. But we have supposed $E$ to be in $I$, so $\phi^+(E) \geq t$ and $\phi > \phi^+(E)$, which is a contradiction. Therefore, $\phi = \phi^+(A) \leq \phi^+(E)$. The second equality follows by a similar argument (just consider the minimum phase of $B$, and argue as above reverting all the arrows). □

*Remark* 2.2.14. The subcategories $\mathscr{P}(> \phi)$ and $\mathscr{P}(\geq \phi)$ are closed under left shift (recall that $\mathscr{P}(\phi+1) = \mathscr{P}(\phi)[1]$ by definition of slicing), and it is very easy to verify that they decompose objects: indeed, fix $\phi \in \mathbb{R}$ and consider a nonzero $E \in \mathscr{D}$. If $\phi^-(E) \leq \phi^+(E) < \phi$, then the triangle $0 \longrightarrow E \longrightarrow E \xrightarrow{+1}$ is distinct by the first axiom and, of course, $E \in \mathscr{P}(\leq \phi)$ and $0 \in \mathscr{P}(\leq \phi)^\perp = \mathscr{P}(> \phi)$. Similarly, if on the other hand, $\phi < \phi^-(E) \leq \phi^+(E)$, then the triangle $E \longrightarrow E \longrightarrow 0 \xrightarrow{+1}$ is distinguished by the first axiom, too, and as above $E \in \mathscr{P}(> \phi)$ while $0 \in \mathscr{P}(> \phi)^\perp = \mathscr{P}(\leq \phi)$. If, on the other hand, we have $\phi^-(E) \leq \phi \leq \phi^+(E)$, then the proof is identical to the one for Proposition 2.1.4 (the part we apply octahedron repeatedly). Indeed, if $E \notin \mathscr{P}(> \phi)$, $E \notin \mathscr{P}(\leq \phi)$, then in the HN filtration of $E$ some of the phases of the semistable quotients are greater than $\phi$, the others are lesser or equal than $\phi$. Consider the maximum index $i$ such that the semistable quotient $A_i \in \mathscr{P}(\phi_i)$ and $\phi_i > \phi$, and complete to a DT the map $E_i \longrightarrow E$. The fact that its cone is in $\mathscr{P}(\leq \phi)$ follows exactly as in Proposition 2.1.4 . Therefore, the pair $(\mathscr{P}(> \phi), \mathscr{P}(\leq \phi))$ defines a t-structure for each $\phi \in \mathbb{R}$ (obviously, the same holds for the pair $(\mathscr{P}(\geq \phi), \mathscr{P}(< \phi))$). The hearts of these t-structures are respectively:

$$\mathscr{H}_{>\phi} = \mathscr{P}(> \phi) \cap \mathscr{P}(> \phi)^\perp[1] = \mathscr{P}(> \phi) \cap \mathscr{P}(\leq \phi + 1) = \mathscr{P}((\phi, \phi+1]) \ ;$$

$$\mathscr{H}_{\geq\phi} = \mathscr{P}(\geq \phi) \cap \mathscr{P}(\geq \phi)^\perp[1] = \mathscr{P}(\geq \phi) \cap \mathscr{P}(< \phi + 1) = \mathscr{P}([\phi, \phi+1)) \ .$$

and, conventionally, the heart of the slicing $\mathscr{P}$ is the category $\mathscr{P}((0,1])$

As we already know, the heart of a t-structure on a triangulated category is always abelian. Therefore, for every $\phi \in \mathbb{R}$, the categories $\mathscr{P}([\phi, \phi+1))$ and $\mathscr{P}((\phi, \phi+1])$ are abelian subcategories. We now wonder if this holds for the subcategories $\mathscr{P}(I)$, too, where $I$ is an arbitrary interval. The answer is that these subcategories are in general not abelian, but if $\ell(I) \leq I$ they are *quasi-abelian.* Recall first that if $\mathscr{A}$ is an additive category and $f \in \mathrm{Hom}_{\mathscr{A}}(A, B)$ , $A, B \in \mathscr{A}$ is a morphism, then $f$ is said to be *strict* if $\mathrm{Im} f \cong \mathrm{Coim} f$. In an abelian category every morphism is clearly strict. We are now ready to give the definition of quasi-abelian category.

**Definition 2.2.15.** Let $\mathscr{A}$ be an additive category. Then $A$ is *quasi-abelian* if

1. for any morphism $f \in \mathrm{Mor}(\mathscr{A})$ we have that $\mathrm{Ker} f, \mathrm{Coker} f \in \mathrm{Ob}(\mathscr{A})$;
2. the class of strict epimorphism is closed under pullback and the class of strict monomorphism is closed under pushout.

Notice that the existence of pullbacks and pushouts follows from the existence of kernels and cokernels. Indeed, the first condition and the fact that $\mathscr{A}$ is additive guarantee the esistence of equalizers and coequalizers: we recall that, given $A, B$

in $\mathscr{A}$ and $f, g \in \mathrm{Hom}_{\mathscr{A}}(A, B)$, the equalizer $\mathrm{Eq}(f, g)$ of $f$ and $g$ is a couple $(E, eq)$ where $E \in \mathrm{Ob}(\mathscr{A})$ and $eq \in \mathrm{Hom}_{\mathscr{A}}(E, A)$ such that for any couple $(O, m)$ with $O \in \mathrm{Ob}(\mathscr{A})$, $m \in \mathrm{Hom}_{\mathscr{A}}(O, A)$ and $f \circ m = g \circ m$, the morphism $m$ factors through $eq$, i.e. there exists $m' \in \mathrm{Hom}_{\mathscr{A}}(O, E)$ s.t. $eq \circ m' = m$ (to obtain the definition of coequalizer simply revert all the arrows). It is very easy to show that, if $\mathscr{A}$ is additive and the kernels and cokernels of $f$ and $g$ are in $\mathscr{A}$, then $\mathrm{Eq}(f, g) = \mathrm{Ker}(f - g)$. Now, given $X, Y, Z \in \mathscr{A}$, a couple of morphisms $f, g$:

$$\begin{array}{ccc} & & X \\ & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow[g]{} & Z \end{array}$$

and the projections $p_X, p_Y$ from the product $X \times Y$ on the two factors, there is a diagram which is not commutative:

$$\begin{array}{ccc} X \times Y & \xrightarrow{p_X} & X \\ {\scriptstyle p_Y}\downarrow & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow[g]{} & Z \end{array} .$$

It is quite a standard exercise to show that pullback of $X$ and $Y$ over $Z$ is simply the triple, given by an object and two projection respectively on $X$ and $Y$, which makes the diagram commute, i.e. $X \times_Z Y = \mathrm{Eq}(f \circ p_X, g \circ p_Y)$.

## 2.3 Stability conditions on triangulated categories

We are now ready to give the key definition.

**Definition 2.3.1.** Let $\mathscr{D}$ be a triangulated category. A *stability condition* on $\mathscr{D}$ consists of a couple $(Z, \mathscr{P}) = \sigma$, where:

1. $Z$ is a group homomorphism $Z : K(\mathscr{D}) \longrightarrow \mathbb{C}$ ,[7]

2. $\mathscr{P}$ is a slicing on $\mathscr{D}$

such that any nonzero object in $\mathscr{P}(\phi)$ for some $\phi \in \mathbb{R}$ is sent by $Z$ to a point whose phase is exactly $\phi$, i.e. $Z(E) = m(E)e^{i\pi\phi}$ for some $m(E) \in \mathbb{R}_{>0}$.

The group homomorphism $Z$ is called *central charge* of the stability condition, and any nonzero $E \in \mathscr{P}(\phi)$ for some $\phi \in \mathbb{R}$ is said to be semistable of phase $\phi$ (the stable object are just the *simple* ones).

**Lemma 2.3.2.** *If $\sigma = (Z, \mathscr{P})$ is a stability condition on a triangulated category $\mathscr{D}$, then the subcategories $\mathscr{P}(\phi) \subset \mathscr{D}$ are abelian for each $\phi \in \mathbb{R}$.*

[7] The definition of Grothendieck group for triangulated categories is very similar to the one for abelian categories: the difference is just that we ask DTs instead of short exact sequences to split.

*Proof.* The subcategory $\mathscr{P}(\phi)$ is full in $\mathscr{D}$, so in particular it is full in $\mathscr{H} := \mathscr{H}_{>\phi-1} = \mathscr{P}((\phi-1,\phi])$, which we know to be abelian. It then suffices to show that if $E, F \in \mathscr{P}(\phi)$ and $f \in \mathrm{Hom}_{\mathscr{P}(\phi)}(E,F) \cong \mathrm{Hom}_{\mathscr{H}}(E,F)$ is a morphism, then $\mathrm{Ker} f$ and $\mathrm{Coker} f$, which we know to be in $\mathscr{H}$, are actually in $\mathscr{P}(\phi)$. Take a short exact sequence $0 \longrightarrow A \longrightarrow E \longrightarrow B \longrightarrow 0$ in $\mathscr{H}$: Lemma 2.2.13 tells us that $\phi^+(A) \leq \phi^+(E) = \phi$ and $\phi = \phi^-(E) \leq \phi^-(B)$. But then $\phi^-(B)$ must be exactly $\phi$, because the phases of nonzero object of $\mathscr{H}$ are upperly bounded by $\phi$, so $\phi^-(B) = \phi^+(B) = \phi$ and $B \in \mathscr{P}(\phi)$. But then $\phi^+(A) = \phi^-(A) = \phi$, because the short exact sequences in the Grothendieck group split and $Z$ is additive, so $Z(E) = Z(A) + Z(B)$. In particular, if $f \in \mathrm{Hom}_{\mathscr{P}(\phi)}(E,F)$, with $E, F \in \mathscr{P}(\phi)$, there are two short exact sequences which are associated to $f$:

$$0 \longrightarrow \mathrm{Ker} f \longrightarrow E \longrightarrow \mathrm{Coim} f \longrightarrow 0$$
$$0 \longrightarrow \mathrm{Im} f \longrightarrow F \longrightarrow \mathrm{Coker} f \longrightarrow 0\ ,$$

therefore $\mathrm{Ker} f, \mathrm{Coker} f, \mathrm{Im} f, \mathrm{Coim} f$ are in $\mathscr{P}(\phi)$. Moreover, $\mathrm{Im} f \cong_{\mathscr{P}(\phi)} \mathrm{Coim} f$ because they are isomorphic in $\mathscr{H}$, which is abelian, and $\mathscr{P}(\phi) \subset \mathscr{H}$ is full. Then $\mathscr{P}(\phi)$ is abelian, too. ☐

**Definition 2.3.3.** The *mass* of $E$ is:

$$m_\sigma(E) = \sum_i |Z(A_i)|$$

where the $A_i$'s are the semistable quotient in the H-N filtration of $E$. By the triangular inequality we get:

$$m_\sigma(E) = \sum_i |Z(A_i)| \geq \left|\sum_i Z(A_i)\right| = |Z(E)|.$$

**Notation 2.3.4.** Let $\sigma = (Z, \mathscr{P})$ be a stability condition on $\mathscr{D}$. We will sometimes write $\phi^\pm_\sigma(E)$ for $\phi^\pm_{\mathscr{P}}(E)$.

**Proposition 2.3.5.** *To give a stability condition on a triangulated category $\mathscr{D}$ is equivalent to giving a bounded t-structure on $\mathscr{D}$ and a stability function on its heart with the Harder-Narasimhan property.*

*Proof.* First suppose $\sigma = (Z, \mathscr{P})$ is a stability condition on $\mathscr{D}$. Then we already know that the category $\mathscr{P}(>0)$ defines a bounded t-structure on $\mathscr{D}$, and that its heart $\mathscr{P}((0,1])$ verifies the Harder-Narasimhan condition (remember Proposition 2.1.6: the boundedness conditions on phases make $\mathscr{P}((0,1])$ verify the hypotesis). Conversely, let $\mathscr{H}$ be the heart of a bounded t-structure on $\mathscr{D}$, and let $Z$ be a stability function on $\mathscr{H}$ with the H-N property. First notice that $K(\mathscr{D}) \cong K(\mathscr{H})$: indeed, any class $[E] \in K(\mathscr{D})$ can be written as the sum $[E] = \sum [A_i]$, where the $A_i$'s are the semistable quotient in the H-N filtration of $E$. The uniqueness of this sum is given by the uniqueness of the H-N filtration. Therefore, the stability condition $Z : K(\mathscr{H}) \longrightarrow \mathbb{C}$ automatically defines the central charge $Z : K(\mathscr{D}) \longrightarrow \mathbb{C}$ of a stability condition on $\mathscr{D}$. The $Z$-semistable object define the slicing $\mathscr{P}$: just set

$$\mathscr{P}(\phi)_{0<\phi\leq 1} := \{Z-\text{semistable objects of phase } \phi\}.$$

Roughly speaking, we are choosing $\mathscr{P}$ so that $\mathscr{H} = \mathscr{P}((0,1])$. Let us check it is actually a slicing.

**a)** We set $\mathscr{P}(\phi+1) := \mathscr{P}(\phi)[1]$ to extend the definition of $\mathscr{P}(\phi)$ to any $\phi \in \mathbb{R}$, so the first condition is automatically satisfied.

**b)** Take $A, B$ in $\mathscr{P}(\phi), \mathscr{P}(\psi)$ respectively. We need to distinguish between two cases:

1. $\phi, \psi$ are both in $(n, n+1]$ for some $n \in \mathbb{Z}$. Then $A, B$ are in $\mathscr{H}[n]$, and we have:

$$\mathrm{Hom}_{\mathscr{H}[n]}(A,B) \cong \mathrm{Hom}_{\mathscr{D}}(A,B) \cong \mathrm{Hom}_{\mathscr{D}}(A[-n],B[-n]) \cong \mathrm{Hom}_{\mathscr{H}}(A[-n],B[-n]) = 0$$

where the first and the third isomorphism hold because the subcategory $\mathscr{H}$ and is full, the second because the shift and its inverse are autoequivalences, and in particular fully faithful, while $\mathrm{Hom}_{\mathscr{H}}(A[-n],B[-n]) = 0$ because of the properties of stability functions on abelian categories. Indeed, now $A[-n]$ and $B[-n]$ are "old style" semistable object (i.e., with respect to a stability function on an abelian category), so just consider that the phase of $A[-n]$ is greater than the phase of $B[-n]$ (if $\phi > \psi \in (n, n+1]$, then obviously $\phi - n > \psi - n \in (0,1]$) and apply Proposition 2.1.4.

2. $\phi \in (n, n+1]$, $\psi \in (m, m+1]$ for some $n > m$. Then just apply Proposition 2.2.7: $\mathscr{H}$ is the heart of a bounded t-structure on $\mathscr{D}$, $A \in \mathscr{H}[n]$, $B \in \mathscr{H}[m]$ with $n > m$, therefore condition 1) says exactly that $\mathrm{Hom}_{\mathscr{D}}(A,B) = 0$.

**c)** We want to find the HN filtration of a nonzero $E \in \mathscr{D}$. In order to do this, we are going to combine the filtration we have inherited from Proposition 2.1.4 with the HN filtration we have on $\mathscr{H}$ as an abelian category. Let $A_i, i = 1, ..., n$ be the factors of the Harder-Narasimhan filtration, i.e. $A_i \in \mathscr{H}_{k_i},\ k_1 > k_2 > ... > k_n$. Each $A_i$, opportunely shifted, has a HN filtration, in the sense of abelian categories, due to the stability function we have on $\mathscr{H}$. Let $0 \subset A_{i1} \subset ... \subset A_{im_i} = A_i[-k\check{\ }i]$, where $A_{ij}/A_{i,j-1} = B_{ij} \in \mathscr{P}(\phi_{ij}), \phi_{ij} \in (0,1]$, be that filtration. Thus we have

$$0 \longrightarrow A_{11}[k_1] \longrightarrow ... \longrightarrow A_{1m_1}[k_1] = A_1 = E_1.$$

If we set $F_i := A_{1i}[k_1],\ i = 1, ..., m_1$ we find the first $m_1$ factors of the HN filtration. Its semistable factors are $B_{ij}[k_1] \in \mathscr{P}(\phi_{ij} + k_1)$. Now we need to go on. By intuition, we would like to have something like the following:

$$E_1 \longrightarrow ? \longrightarrow B_{21}[k_2] \xrightarrow{+1}$$

i.e., an element whose quotient by $E_1$ is exactly the first semistable quotient of the Harder-Narasimhan filtration of $A_2[-k_2]$, opportunely shifted. By intuition again, it can be built as the cone of a map $B_{21}[k_2-1] \longrightarrow E_1$ (notice that

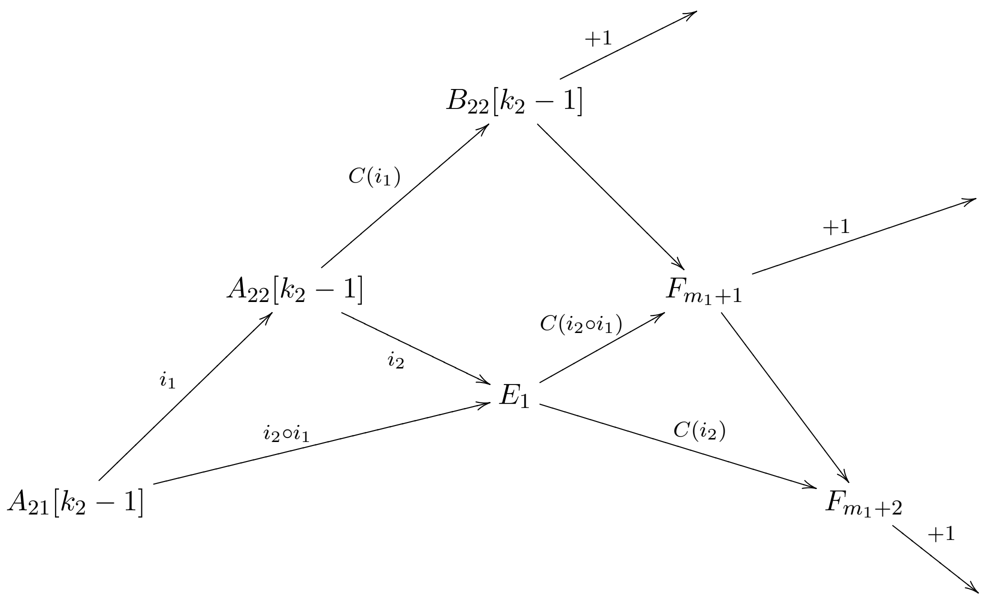
$B_{21} \cong A_{21}$). The first question which arises is: does such a map exist? It does: indeed, consider the composition:

$$A_{21}[k_2] \longrightarrow A_2[k_2] \longrightarrow E[1],$$

where the first map is just the shifted inclusion $A_{21} \subset A_{22} \subset ... \subset A_{2m_2} = A_2$, while the second one is the shift map in the second triangle of the HN filtration of $E$ we had by Proposition 2.1.4:

$$E_1 \longrightarrow E_2 \longrightarrow A_2 \xrightarrow{+1} E_1[1].$$

We have thus obtained a map $A_{21}[k_2] \longrightarrow E[1]$ and, therefore, a map $A_{21}[k_2 - 1] \longrightarrow E$. Calling $F_{m_1+1}$ the cone of this map we find the $m_1$+1-factor. Now we can iterate: define $F_{m_1+i}$ as the cone of the map $A_{2i}[k_2 - 1] \longrightarrow E_1$, which, as above, is the composition of the shifted inclusion $A_{2i} \subset A_{2,i+1} \subset ... \subset A_2$ and the shift map $A_2 \longrightarrow E_[1]$. Now, a *deus ex machina* is necessary to show us that everything works: we apply the octahedron axiom to the map $A_{21}[k_2 - 1] \longrightarrow E_1$, which is itself the composition $A_{21}[k_2 - 1] \xrightarrow{i_1} A_{22}[k_2 - 1] \xrightarrow{i_2} E_1$. Look what happens:




the triangle $B_{22}[k_2 - 1] \longrightarrow F_{m_1+1} \longrightarrow F_{m_1} \xrightarrow{+1}$ is exact, so we get that the cone of the map $F_{m_1+1} \longrightarrow F_{m_1+2}$ is exactly $B_{22}[k_2] \in \mathscr{P}(\phi_{22} + k_2)$! The same argument applied to the map $A_{2i}[k_2 - 1] \longrightarrow E_1$, considered as the composition $A_{2i}[k_2 - 1] \longrightarrow A_{2,i+1}[k_2 - 1] \longrightarrow E_1$, shows us that the quotient are exactly the $B_{2i}[k_2 - 1]$'s. An iteration on the $A_j$'s, $j = 3, ..., n$, builds up the HN filtration we are looking for, completing thus the proof.

□

*Remark* 2.3.6. Notice that what the Proposition states implies that a stability condition can be give either as the pair $(Z, \mathscr{P})$ of Definition 2.3.1, or as a pair $(Z, \mathscr{A})$, where $Z$ is the usual group homomorphism and $\mathscr{A}$ is the heart of a bounded t-structure on the category $\mathscr{D}$, and the objects belonging to $\mathscr{A}$ are sent by $Z$ to the upper-half plane $\mathbb{H}$. This will be particularly useful in the following chapters, when we will build stability conditions on the derived category of a K3 surface.

Let us give an example of how this proposition can be applied.

**Example 2.3.7.** Let $X$ be a projective nonsingular curve over an algebraically closed field $k$, with $char k = 0$. We already know there is a "classical" stability function on $\mathscr{A} = \mathrm{Coh}(X)$[8], i.e., for each nonzero $\mathscr{F} \in \mathscr{A}$, we set:

$$Z(\mathscr{F}) := -\deg(\mathscr{F}) + i\mathrm{rank}(\mathscr{F}) \ .$$

By applying the proposition, we get a stability condition on $\mathscr{D}^b(\mathscr{A})$, where $\mathscr{A}$ is the heart of the trivial t-structure.

## 2.4 The space of stability conditions

Let $\mathscr{D}$ be a triangulated category.

**Definition 2.4.1.** A slicing $\mathscr{P}$ on a triangulated category $\mathscr{D}$ is said to be *locally finite* if there exists $\eta \in \mathbb{R}_{>0}$ such that for each $t \in \mathbb{R}$ the quasi-abelian category $\mathscr{P}((t-\eta, t+\eta)) \subset \mathscr{D}$ is of finite length[9]. A stability condition $\sigma = (Z, \mathscr{P})$ is locally finite if the slicing $\mathscr{P}$ is.

**Definition 2.4.2.** We define the following sets:

$$\mathrm{Slice}(\mathscr{D}) = \{\text{locally finite slicings on } \mathscr{D}\},$$

$$\mathrm{Stab}(\mathscr{D}) = \{\text{locally finite stability conditions on } \mathscr{D}\}.$$

We will later notice that almost everything we do works well even without the locally finiteness, but it is useful to avoid pathological cases. What we want to do is to give the set $\mathrm{Stab}(\mathscr{D})$ the structure of a topplogical space. First of all, we will need to consider the inclusion:

$$\mathrm{Stab}(\mathscr{D}) \subset \mathrm{Hom}_{\mathbb{Z}}(K(\mathscr{D}), \mathbb{C}) \times \mathrm{Slice}(\mathscr{D}).$$

Put a generalized metric on the set $\mathrm{Slice}(\mathscr{D})$ by setting

$$d(\mathscr{P}, \mathscr{Q}) := \sup_{0 \neq E \in \mathscr{D}} \{|\phi^-_{\mathscr{P}}(E) - \phi^-_{\mathscr{Q}}(E)|, |\phi^+_{\mathscr{P}}(E) - \phi^+_{\mathscr{Q}}(E)|\} \subset [0, +\infty] \ .$$

Let us check it is actually a generalized metric:

1. $d(\mathscr{P}, \mathscr{Q}) = 0 \ \Rightarrow \ |\phi^-_{\mathscr{P}}(E) - \phi^-_{\mathscr{Q}}(E)| = |\phi^+_{\mathscr{P}}(E) - \phi^+_{\mathscr{Q}}(E)| = 0$ for all nonzero $E \in \mathscr{D} \ \Rightarrow \ \phi^-_{\mathscr{P}}(E) = \phi^-_{\mathscr{Q}}(E)$ , $\phi^+_{\mathscr{P}}(E) = \phi^+_{\mathscr{Q}}(E)$ for all nonzero $E \in \mathscr{D} \ \Rightarrow \ \mathscr{P} = \mathscr{Q}$. The converse is obvious.

[8] We will later treat this example more accurately.

[9] We recall that a category is of finite length if every object is both artinian and noetherian.

2. It is obviously symmetrical, because the absolute value in $\mathbb{R}$ is.

3. $d(\mathscr{P},\mathscr{Q}) = \sup_{0\neq E\in\mathscr{D}} \{|\phi^-_{\mathscr{P}}(E) - \phi^-_{\mathscr{Q}}(E)|, |\phi^+_{\mathscr{P}}(E) - \phi^+_{\mathscr{Q}}(E)|\} =$
$= \sup_{0\neq E\in\mathscr{D}} \{|\phi^-_{\mathscr{P}}(E) - \phi^-_{\mathscr{R}}(E) + \phi^-_{\mathscr{R}}(E) - \phi^-_{\mathscr{Q}}(E)|, |\phi^+_{\mathscr{P}}(E) - \phi^+_{\mathscr{R}}(E) + \phi^+_{\mathscr{R}}(E) -$
$\phi^+_{\mathscr{Q}}(E)|\} \leq$
$\leq \sup_{0\neq E\in\mathscr{D}} \{|\phi^-_{\mathscr{P}}(E) - \phi^-_{\mathscr{R}}(E)| + |\phi^-_{\mathscr{R}}(E) - \phi^-_{\mathscr{Q}}(E)|, |\phi^+_{\mathscr{P}}(E) - \phi^+_{\mathscr{R}}(E)| + |\phi^+_{\mathscr{R}}(E) -$
$\phi^+_{\mathscr{Q}}(E)|\} \leq$
$\leq \sup_{0\neq E\in\mathscr{D}} \{|\phi^-_{\mathscr{P}}(E)-\phi^-_{\mathscr{R}}(E)|, |\phi^+_{\mathscr{P}}(E)-\phi^+_{\mathscr{R}}(E)|\} + \sup_{0\neq E\in\mathscr{D}} \{|\phi^-_{\mathscr{R}}(E)-\phi^-_{\mathscr{Q}}(E)|, |\phi^+_{\mathscr{R}}(E)-$
$\phi^+_{\mathscr{Q}}(E)|\} =$
$= d(\mathscr{P},\mathscr{R}) + d(\mathscr{R},\mathscr{Q}).$

It will sometimes be useful to write this metric in a different way:

**Lemma 2.4.3.** *If $\mathscr{P},\mathscr{Q}$ are in* Slice$(\mathscr{D})$*, then*

$$d'(\mathscr{P},\mathscr{Q}) := \inf\{\varepsilon \in \mathbb{R}_{\geq 0} \mid \mathscr{Q}(\phi) \subset \mathscr{P}([\phi-\varepsilon,\phi+\varepsilon]) \ \forall\phi\in\mathbb{R}\} = d(\mathscr{P},\mathscr{Q}) \ .$$

*Proof.* First consider that if $d(\mathscr{P},\mathscr{Q}) \leq \varepsilon$, then$|\phi^-_{\mathscr{P}} - (E) - \phi^-_{\mathscr{Q}}(E)| \leq \varepsilon$, $|\phi^+_{\mathscr{P}} - (E) - \phi^+_{\mathscr{Q}}(E)| \leq \varepsilon$; therefore if $0 \neq E \in \mathscr{Q}(\phi)$, then $\phi-\varepsilon \leq \phi^-_{\mathscr{P}}(E) \leq \phi^+_{\mathscr{P}}(E) \leq \phi+\varepsilon$. This means that $\mathscr{Q}(\phi) \subset \mathscr{P}([\phi-\varepsilon,\phi+\varepsilon])$. For the converse, suppose $d'(\mathscr{P},\mathscr{Q}) \leq \varepsilon$. Take a nonzero $E \in \mathscr{D}$. If $E \in \mathscr{Q}(\leq\phi)$, then clearly $E \in \mathscr{P}(\leq \psi+\varepsilon)$. If $E \notin \mathscr{Q}(\leq\psi)$, then there is a nonzero object $A \in \mathscr{Q}(\phi)$, with $\phi > \psi$, and a nonzero map $A \longrightarrow E$. Since $A \in \mathscr{P}([\phi-\varepsilon,\phi+\varepsilon])$, then $E$ can't be in $\mathscr{P}(\leq \psi-\varepsilon)$. Therefore $|\phi^+_{\mathscr{P}} - \phi^+_{\mathscr{Q}}| \leq \varepsilon$. A symmetrical argument shows that $d(\mathscr{P},\mathscr{Q}) \leq \varepsilon$.

□

Now consider the complex vector space $\mathrm{Hom}_{\mathbb{Z}}(K(\mathscr{D}),\mathbb{C})$. We can associate a generalized to each $\sigma = (Z,\mathscr{P}) \in \mathrm{Stab}(\mathscr{D})$:

$$||\cdot||_\sigma : \quad \mathrm{Hom}_{\mathbb{Z}}(K(\mathscr{D}),\mathbb{C}) \longrightarrow [0,+\infty]$$

$$U \longmapsto ||U||_\sigma = \sup_{E\ \sigma-ss} \frac{|U(E)|}{|Z(E)|} \ .$$

Let us check it is actually a generalized norm:

1. Obviously $||U||_\sigma \geq 0$ for each $U \in \mathrm{Hom}_{\mathbb{Z}}(K(\mathscr{D}),\mathbb{C})$ (the module function is always positive), while if $||U||_\sigma = 0$, then for each semistable $E$ the module $|U(E)| = 0$; therefore, $U = 0$ itself, because the Grothendieck group is generated by the class of semistable objects;

2. $||nU||_\sigma = \sup_{E\ \sigma-ss} \frac{|nU(E)|}{|Z(E)|} = \sup_{E\ \sigma-ss} \frac{|n||U(E)|}{|Z(E)|} = |n|||U||_\sigma$;

3. $||U+V||_\sigma = \sup_{E\ \sigma-ss} \frac{|U(E)+V(E)|}{|Z(E)|} \leq \sup_{E\ \sigma-ss} \frac{|U(E)|+|V(E)|}{|Z(E)|} = \sup_{E\ \sigma-ss} (\frac{|U(E)|}{|Z(E)|} + \frac{|V(E)|}{|Z(E)|}) \leq$
$\leq \sup_{E\ \sigma-ss} \frac{|U(E)|}{|Z(E)|} + \sup_{E\ \sigma-ss} \frac{|V(E)|}{|Z(E)|} = ||U||_\sigma + ||V||_\sigma$.

We are now ready to put a topology on the set $\mathrm{Stab}(\mathscr{D})$, using both the generalized metric and the generalized norm we have put on $\mathrm{Slice}(\mathscr{D})$ and $\mathrm{Hom}_{\mathbb{Z}}(K(\mathscr{D}),\mathbb{C})$ respectively. For each $\sigma \in \mathrm{Stab}(\mathscr{D})$ and for each $\varepsilon \in (0, \frac{1}{8})$, we define the open ball of center $\sigma$ and radius $\varepsilon$ in the following way:

$$B_\varepsilon(\sigma) = \{\tau = (W, \mathscr{Q}) \in \mathrm{Stab}(\mathscr{D}) \mid ||W - Z||_\sigma < \sin(\pi\varepsilon)\ ,\ d(\mathscr{P}, \mathscr{Q}) < \varepsilon\} \subset \mathrm{Stab}(\mathscr{D}).$$

*Remark* 2.4.4. Notice that the condition $||W - Z||_\sigma < \sin(\pi\varepsilon)$ simply means that, if $E$ is semistable with respect to $\sigma$, then the distance between the phases of $Z(E)$ and $W(E)$ must be lesser than $\varepsilon$.

**Lemma 2.4.5.** *If $\tau = (W, \mathscr{Q}) \in B_\epsilon(\sigma)$, then there exist $k_1, k_1 \in \mathbb{R}_{>0}$ such that*

$$k_1||U||_\sigma < ||U||_\tau < k_2||U||_\sigma\ .$$

*Proof.* For each $\sigma \in \mathrm{Stab}(\mathscr{D})$, $\eta \in [0, \frac{1}{2})$, the following holds for each nonzero $E \in \mathscr{D}$ and $U \in \mathrm{Hom}_{\mathbb{Z}}(K(\mathscr{D}),\mathbb{C})$:

$$|\phi^+_\sigma - \phi^-_\sigma| < \eta\ ,\ |U(E)| < \frac{||U||_\sigma}{\cos(\pi\eta)}|Z(E)|.$$

Indeed, we can write $Z(E)$ as the sum of the $Z(A_i)$'s, where the $A_i$'s are the semistable quotients in the HN filtration of $E$. Therefore:

$$Z(E) = \sum_i Z(A_i) \quad \Rightarrow \quad |Z(E)| = \sum_i |Z(A_i)| \cos(\pi\phi_i)$$

where $\phi_i$ is the angle between $Z(E)$ and $Z(A_i)$. But we have that $|\phi^+_\sigma(E) - \phi^-_\sigma(E)| < \eta$, so it follows that for each $i$, $\phi_i \le \eta$ and $\cos(\pi\phi_i) \ge \cos(\pi\eta)$. Therefore:

$$|Z(E)| = \sum_i |Z(A_i)| \cos(\pi\phi_i) > \sum_i |Z(A_i)| \cos(\pi\eta) \quad \Rightarrow \quad \frac{|Z(E)|}{\cos(\pi\eta)} > \sum_i |Z(A_i)|.$$

Moreover, for each $A_i$, we have that $||U||_\sigma > \frac{|U(A_i)|}{|Z(A_i)|}$ (it is the sup over the class of $\sigma$-semistable objects, and the $A_i$ are semistable). It follows that $|U(A_i)| < ||U||_\sigma |Z(A_i)|$, so $\sum_i |U(A_i)| < ||U||_\sigma \sum_i |Z(A_i)|$. But $|U(E)| \le \sum_i |U(A_i)|$ and $\sum_i |U(A_i)| < ||U||_\sigma \frac{|Z(E)|}{\cos(\pi\eta)}$, therefore $|U(E)| < \frac{||U||_\sigma}{\cos(\pi\eta)}|Z(E)|$. Now, if $\tau = (W, \mathscr{Q}) \in B_\varepsilon(\sigma)$, we can apply this inequality to $U = W - Z$ and $\eta = 2\varepsilon$, obtaining

$$|W(E) - Z(E)| \le \frac{||W - Z||_\sigma}{\cos(2\pi\varepsilon)} < \frac{\sin(\pi\varepsilon)}{\cos(2\pi\varepsilon)}|Z(E)|$$

for each $E \in \mathscr{D}$ $\tau$-semistable. This means that:

$$|W(E)| - |Z(E)| \le |W(E) - Z(E)| < \frac{\sin(\pi\varepsilon)}{\cos(2\pi\varepsilon)}|Z(E)| \ \Rightarrow$$

$$\Rightarrow |W(E)| < (1 + \frac{\sin(\pi\varepsilon)}{\cos(2\pi\varepsilon)})|Z(E)| \quad \Rightarrow \quad |W(E)| < k|Z(E)|,$$

where $k := 1 + \frac{\sin(\pi\varepsilon)}{\cos(2\pi\varepsilon)}$. Therefore:

$$\frac{1}{|W(E)|} > \frac{1}{k}\frac{1}{|Z(E)|};$$

$$\frac{|U(E)|}{|W(E)|} > \frac{1}{k}\frac{|U(E)|}{|Z(E)|} > \frac{1}{k}||U||_\sigma.$$

Moreover:

$$|U(E)| < \frac{||U||_\tau}{\cos(\pi\eta)}|W(E)|;$$

$$\frac{|U(E)|}{|W(E)|} < \frac{||U||_\tau}{\cos(\pi\eta)}$$

and finally:

$$\frac{||U||_\sigma}{\cos(\pi\eta)} > \frac{|U(E)|}{|W(E)|} > \frac{1}{k}\frac{|U(E)|}{|Z(E)|} > \frac{1}{k}||U||_\sigma$$

which means that

$$||U||_\tau > \frac{\cos(\pi\eta)}{k}||U||_\sigma \;\Rightarrow\; k_1||U||_\sigma < ||U||_\tau$$

where $k_1 := \frac{\cos(\pi\eta)}{k}$. The other inequalty follow with the same argument, and simply switching $Z$ and $W$.

□

We are now ready to prove that the open balls $B_\varepsilon(\sigma)$, with $\sigma$ varying in Stab($\mathscr{D}$) and $\varepsilon$ in $(0, \frac{1}{8})$, form a basis for a topology on Stab($\mathscr{D}$). We need to show that:

1. $\bigcup_{\varepsilon,\sigma} B_\varepsilon(\sigma) = \mathrm{Stab}(\mathscr{D})$.
2. If $\tau \in B_\varepsilon(\sigma)$, then there exists $\eta > 0$ such that $B_\eta(\tau) \subset B_\epsilon(\sigma)$.

The first statement is obvious (just think that each $\sigma \in \mathrm{Stab}(\mathscr{D})$ is the center of an open ball). The second one follows from the lemma above. Indeed, consider $\tau = (W, \mathscr{Q}) \in B_\varepsilon(\sigma)$.

1. It holds that $d(\mathscr{P}, \mathscr{Q}) < \varepsilon$. We want to show that there exists $r > 0$ such that $d(\mathscr{Q}, \mathscr{R}) < r \;\Rightarrow\; d(\mathscr{P}, \mathscr{R}) < \varepsilon$. Take $2r = \min\{d(\mathscr{P}, \mathscr{Q}), \varepsilon - d(\mathscr{P}, \mathscr{Q})\}$. By triangle inequality, we get that $d(\mathscr{P}, \mathscr{R}) \leq d(\mathscr{P}, \mathscr{Q}) + d(\mathscr{Q}, \mathscr{R})$.

   $d(\mathscr{P}, \mathscr{Q}) \in [0, \frac{\varepsilon}{2}]$. Then:

$$2r = d(\mathscr{P}, \mathscr{Q}) \;\Rightarrow\; r = \frac{d(\mathscr{P}, \mathscr{Q})}{2};$$

$$d(\mathscr{P}, \mathscr{R}) \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{4} = \frac{3}{4}\varepsilon < \varepsilon.$$

$d(\mathscr{P},\mathscr{Q}) \in (\frac{\varepsilon}{2},\varepsilon)$. Then:

$$2r = \varepsilon - d(\mathscr{P},\mathscr{Q}) \;\Rightarrow\; r = \frac{\varepsilon - d(\mathscr{P},\mathscr{Q})}{2};$$

$$d(\mathscr{P},\mathscr{R}) \le d(\mathscr{P},\mathscr{Q}) + \frac{\varepsilon - d(\mathscr{P},\mathscr{Q})}{2} = \frac{d(\mathscr{P},\mathscr{Q})+\varepsilon}{2} < \frac{2\varepsilon}{2} = \varepsilon.$$

2. $||W-Z||_\sigma < \sin\pi\varepsilon$. We want to show that there exists $s>0$ such that if $||W-X||_\tau < \sin\pi s \;\Rightarrow\; ||Z-X||_\sigma < \sin\pi\varepsilon$, with $X \in \mathrm{Hom}_{\mathbb{Z}}(K(\mathscr{D}),\mathbb{C})$. Again by triangle inequality, we get that $||Z-X||_\sigma \le ||W-Z||_\sigma + ||W-X||_\sigma$. Therefore, using the Lemma, if $k$ is a suitable constant:

$$||Z-X||_\sigma \le ||W-Z||_\sigma + ||W-X||_\sigma < ||W-Z||_\sigma + k||W-X||_\tau <$$
$$< \sin\pi\varepsilon + k\sin\pi s.$$

Now, we want $\sin\pi\varepsilon + k\sin\pi s < \sin\pi\varepsilon$, therefore it must be $\sin\pi s < 0$ (remind that all the constants in the Lemma are positive). Does such an $s$ exist? Of course it does: for example, taking $s \in (1,2)$ will work.

We put on $\mathrm{Stab}(\mathscr{D})$ the topology whose base are the open balls defined above. By the Lemma, the subspace

$$\{U \in \mathrm{Hom}_{\mathbb{Z}}(K(\mathscr{D}),\mathbb{C}) \mid ||U||_\sigma < +\infty\} \subset \mathrm{Hom}_{\mathbb{Z}}(K(\mathscr{D}),\mathbb{C})$$

is locally constant on $\mathrm{Stab}(\mathscr{D})$, i.e. if $||U||_\sigma < +\infty$, then for each $\varepsilon \in (0,\frac{1}{8})$ and $\tau \in B_\varepsilon(\sigma)$ we have that $||U||_\tau < +\infty$, so this holds for each $\tau$ in the same connected component of $\sigma$. Let $\Sigma$ be a connected component of $\mathrm{Stab}(\mathscr{D})$ and set:

$$V(\Sigma) := \{U \in \mathrm{Hom}_{\mathbb{Z}}(K(\mathscr{D}),\mathbb{C}) \mid ||U||_\sigma < +\infty \;,\; \sigma \in \Sigma\}.$$

*Remark* 2.4.6. If $\sigma = (Z,\mathscr{P})$ is in $\Sigma$, then $Z \in V(\Sigma)$: obviously

$$||Z||_\sigma = \sup_{E\ \sigma-ss} \frac{|Z(E)|}{|Z(E)|} = 1.$$

*Remark* 2.4.7. The Lemma says exactly that all these norms are equivalent, i.e., they induce the same topology.

We are now ready to state the key theorem:

**Theorem 2.4.8.** *The map*

$$\mathscr{Z} : \Sigma \longrightarrow V(\Sigma)$$
$$(Z,\mathscr{P}) \mapsto Z \;,$$

*where both $\Sigma$ and $V(\Sigma)$ have the subspace topology, is a local homeomorphism.*

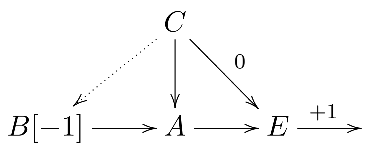
The local injectivity follows from the proposition below:

**Proposition 2.4.9.** *Suppose $\sigma = (Z,\mathscr{P}),\ \tau = (Z,\mathscr{Q}) \in \mathrm{Stab}(\mathscr{D})$. If $d(\mathscr{P},\mathscr{Q}) < 1$, then $\sigma = \tau$.*

*Proof.* Suppose by contradiction that $\sigma \neq \tau$, i.e., considering that the central charge is the same, that $\mathscr{P} \neq \mathscr{Q}$. It means that there exists $\phi \in \mathbb{R}$ and $E \in \mathscr{P}(\phi)$ such that $E \notin \mathscr{Q}(\phi)$. It cannot be $E \in \mathscr{Q}(\geq \phi)$ because the fact that $d(\mathscr{P}, \mathscr{Q}) < 1$ would imply that $E \in \mathscr{Q}([\phi, \phi+1))$, contradicting the fact that $\sigma$ and $\tau$ have the same central charge: indeed, $Z(E) = W(E)$ would mean in particular that $\phi_Z(E) = n\phi_W(E)$ for some $n \in \mathbb{Z}$. For the same reason, it cannot be $E \in \mathscr{Q}(\leq \phi)$. Therefore, there exists a DT

$$A \longrightarrow E \longrightarrow B \xrightarrow{+1}$$

with both $A$ and $B$ nonzero, $A \in \mathscr{Q}((\phi, \phi+1))$, $B \in \mathscr{Q}((\phi-1, \phi])$. It cannot be $A \in \mathscr{P}(\leq \phi)$, because it would be $A \in \mathscr{P}((\phi-1, \phi])$, contradicting the fact that the two stability conditions have the same central charge, as above. So there exists an object $C \in \mathscr{P}(\psi)$, with $\psi > \phi$, and a nonzero map $C \longrightarrow A$. Consider the following diagram:




the composition $C \longrightarrow A \longrightarrow B$ must be zero, because $\psi > \phi$, then the map $C \longrightarrow A$ lifts to $B[-1]$. But $B[-1] \in \mathscr{P}(\leq \phi - 1)$, therefore this is a contradiction. □

The local surjectivity follows from the theorem below:

**Theorem 2.4.10.** *Let $\sigma = (Z, \mathscr{P}) \in \mathrm{Stab}(\mathscr{D})$. Then there exists $\epsilon_0 > 0$ such that if $0 < \epsilon < \epsilon_0$ and $W \in \mathrm{Hom}_{\mathbb{Z}}(K(\mathscr{D}), \mathbb{C})$ satisfies*

$$|W(E) - Z(E)| < \sin \pi\varepsilon |Z(E)|$$

*for each $E$ semistable with respect to $\sigma$, then there exists a slicing $\mathscr{Q} \in \mathrm{Slice}(\mathscr{D})$ such that $\tau = (W, \mathscr{Q})$ is a stability condition and $d(\mathscr{P}, \mathscr{Q}) < \varepsilon$.*

Roughly speaking, the theorem says that if we consider a stability condition, a slight deformation of its central charge and a suitable choice of a slicing lead to a new stability condition. This means that the map in the key theorem is locally surjective. The proof of this theorem is quite long and technical, so we will just give a sketch and skip the details.

*Proof.* **(sketch)**. What we do is simply define a slicing which we imagine to make everything work. Take $\psi \in \mathbb{R}$ and set:

$$\mathscr{Q}(\psi) := \{0 \in \mathscr{D}\} \cup \{E \in \mathscr{D} \mid E\ W\text{−semistable of phase}\psi \text{ in } \mathscr{P}((a,b)) \text{ with } a+\varepsilon \leq \psi \leq b-\varepsilon \text{ for some } \varepsilon > 0\}.$$

We need to show that $\mathscr{Q}$ is actually a slicing and that $(W, \mathscr{Q})$ is a stability condition. The fact that $\mathscr{Q}$ is a slicing follows from the two lemmas below:

**Lemma 2.4.11.** *If $E \in \mathscr{Q}(\psi_1)$, $F \in \mathscr{Q}(\psi_2)$ and $\psi_1 > \psi_2$, then* $\mathrm{Hom}_{\mathscr{D}}(E, F) = 0$

(the strategy of the proof is quite similar to the one used for Proposition 2.1.4, reminding that the category we are dealing with is not abelian, but quasi-abelian).

**Lemma 2.4.12.** *Let $\mathscr{A} = \mathscr{P}((a,b)) \subset \mathscr{D}$ a finite length subcategory such that $0 < b-a < 1-2\epsilon$. Then each nonzero object of $\mathscr{P}((a+2\varepsilon, b-4\varepsilon))$ possesses a HN filtration, whose $W$-semistable factors are objects of $\mathscr{A}$, and moreover $a+\varepsilon \leq \phi_i \leq b-\varepsilon$, where $\phi_i$ is the phase of the i-th semistable factor.*

(this is similar to what we proved for Proposition 2.1.6, but with much more technical details to fix).

The fact that $\tau = (W, \mathscr{Q})$ simply follows from the definition of $\mathscr{Q}$.

□

## 2.5 The natural actions

In this section we will show that, for each triangulated category $\mathscr{D}$, there are two natural actions on the space of stability conditions $\mathrm{Stab}(\mathscr{D})$. These two actions will be very important because they will allow us to describe, partially or totally, the space $\mathrm{Stab}(\mathscr{D})$ where $\mathscr{D}$ is the derived category of some variety.

- An action on the right is provided by the topological group $\tilde{GL}^+(2,\mathbb{R})$. Notice that we can identify $\tilde{GL}^+(2,\mathbb{R}) = \{(T,f) \mid T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2,\ detT > 0$ and $f : \mathbb{R} \longrightarrow \mathbb{R}$ increasing with the property that $f(\phi+1) = f(\phi)+1 \in \mathbb{R}$ such that the induced maps on $S^1 = \mathbb{R}/2\mathbb{Z} = \mathbb{R}^2 \setminus \{0\}/\mathbb{R}_+$ are the same $\}$.

  The reason of this identification is the following: we know that the fundamental group $\pi_1(GL^+(2,\mathbb{R})) = \mathbb{Z}$, as it is homotopic to the product $S^1 \times \mathbb{R}^3$, therefore it is a $\mathbb{Z}$-principal bundle . Obviously there is an action $\tilde{GL}^+(2,\mathbb{R}) \times \mathbb{Z} \longrightarrow GL^+(2,\mathbb{R})$ which preserves the fibers, acting freely and transitively on them. As a $\mathbb{Z}$-principal bundle, it is induced by a continuous map

  $$GL^+(2,\mathbb{R}) \longrightarrow S^1$$
  $$(a,b,c,d) \mapsto \frac{1}{\sqrt{a^2+c^2}}(a,c)$$

  and therefore an element can be seen as $(T,x)$, where $T \in GL^+(2,\mathbb{R})$ and $x$ is the first vector of $T$ normalized. Now we can see $x$ as the initial value of a lift $f : \mathbb{R} \longrightarrow \mathbb{R}$ of a map $\tilde{T} : S^1 \longrightarrow S^1$ (the one induced by T). As $\mathbb{R} \longrightarrow S^1$ is a covering, $f$ is completely determined by $T$ and the initial point $x$.

  Now, the action is given by;

  $$\mathrm{Stab}(\mathscr{D}) \times \tilde{GL}^+(2,\mathbb{R}) \longrightarrow \mathrm{Stab}(\mathscr{D})$$
  $$(Z,\mathscr{P}),(T,f) \mapsto (T^{-1} \circ Z, \mathscr{P}')$$

  where $\mathscr{P}'(\phi) = \mathscr{P}(f(\phi))$.

- The other action is the one provided by the group $\mathrm{Aut}\mathscr{D}$:

  $$\mathrm{Aut}\mathscr{D} \times \mathrm{Stab}(\mathscr{D}) \longrightarrow \mathrm{Stab}(\mathscr{D})$$

$$\Phi \quad , \quad (Z, \mathscr{P}) \mapsto \quad (Z \circ \Phi^{-1}, \Phi(\mathscr{P}))$$

where $\Phi(\mathscr{P})(\phi) = \Phi(\mathscr{P}(\phi))$.

# Chapter 3

# Fourier-Mukai transforms and derived equivalences

Fourier-Mukai transforms are a powerful tool when dealing with derived categories of coherent sheaves over a variety. Functors that are of Fourier-Mukai type behave well with respect to elementary functorial operation: they admit left and right adjoints, the composition of two Fourier-Mukai transforms is again of Fourier-Mukai type and, finally, there are explicit conditions which allow one to decide whether a Fourier-Mukai transform is an equivalence or not. There are two main results in the theory of Fourier-Mukai transforms: the first one, which is due to Orlov (Theorem 3.1.5), states that each exact quivalence between two derived category $\mathscr{D}(X)$ and $\mathscr{D}(Y)$, where X and Y are smooth projective varieties, is isomorphic to an equivalence of Fourier-Mukai type. The second one, which is a corollary of Theorem 3.1.5, gives a description of the group $\mathrm{Aut}\mathscr{D}$ in some particular cases. It states that if $X$ is a smooth projective variety with ample canonical or anticanonical sheaf, then the group $\mathrm{Aut}\mathscr{D}(X)$ is generated by shifts, automorphisms of the variety and twist by line bundles (we will later explain what a twist functor is). For a complete proof of this Theorem, see [6].

Let us now explain some heuristic behind the Fourier-Mukai transforms. First, recall what the classical the Fourier transform is. It is something like this: given a function $f(x)$, the Fourier transform of $f$ is the function $g(y) := \int f(x)e^{2\pi ixy}dx$.

Let us give a quick description of the Fourier-Mukai transform:

1. Given two varieties $X$ and $Y$, and a sheaf $\mathcal{P}$ on $X \times Y$. The sheaf $\mathcal{P}$ sometimes is called the "integral kernel". Take a sheaf $\mathcal{F}$ on $X$. Think of $\mathcal{F}$ as being analogous to the function $f(x)$ in the classical situation. Think of $\mathcal{P}$ as being the analogous, in the classical situation, of some function of $x$ and $y$.

2. Now pull back the sheaf along the projection $q : X \times Y \to X$. Think of the pullback $q^*\mathcal{F}$ as being the analogous of the function $f(x)$, and of $\mathcal{P}$ as being analogous to the function $e^{2\pi ixy}$.

3. Next, take the tensor product $q^*\mathcal{F} \otimes \mathcal{P}$. This is analogous to the function $f(x)e^{2\pi ixy}$.

4. Finally, push down $q^*\mathcal{F} \otimes \mathcal{P}$ along the projection $p : X \times Y \to Y$. The result is the Fourier-Mukai transform of $\mathcal{F}$ — it is $p_*(q^*\mathcal{F} \otimes \mathcal{P})$. This last pushforward step can be thought of as "integration along the fiber": here the fiber direction is the $X$ direction. So in the classical situation it is $g(y) = \int f(x)e^{2\pi ixy}dx$, which is the Fourier transform of $f(x)$.

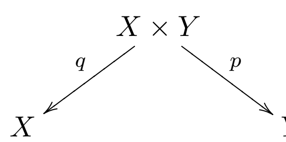
To make all of this rigorous, we have to deal with derived categories of coherent sheaves, not just coherent sheaves. In this context the main difficulty is the pushforward operation. As is well known, the pushforward of a coherent sheaf is not always coherent. But we can use the derived pushfoward instead, at the "price" of having to deal with derived categories.

When $X$ is an abelian variety, $Y$ is the dual abelian variety, and $\mathcal{P}$ is the Poincare line bundle on $X \times Y$, then the Fourier-Mukai transform gives an equivalence of the derived category of coherent sheaves on $X$ with the derived category of coherent sheaves on $Y$. This was proved by Mukai. This is supposed to be analogous to the statement I made about the classical Fourier transform being invertible. In other words the Poincare line bundle is really supposed to be analogous to the function $e^{2\pi ixy}$. A more general choice of $\mathcal{P}$ corresponds to, in the classical situation, so-called integral transforms, i.e. transforms of Fourier type with a different kernel. They do not have, in general, all the good properties of the Fourier transform, but they can be nonetheless studied to provide examples of transforms between functions in $L^p$ spaces. This is probably why $\mathcal{P}$ is called the integral kernel: to recall the kernel of the Fourier-transform. When $X$ is an abelian variety, therefore, the analogies between the classical Fourier transform and the Fourier-Mukai transforms are stronger, and some of the properties which make the Fourier transform a powerful tool in analysis are analogously resembled in the algebraic geometric version. This topic is completely treated in [13].

In this chapter, we will give the definition of the Fourier-Mukai transform, and we will its basic properties and give some interesting examples of how it can be applied.

## 3.1 Definition and first properties

Let $X$ and $Y$ be smooth projective varieties[1], and let




[1] By a variety we mean an integral separated scheme of finite type over an algebraically closed field

be the projection on each of the two factors. To each object $\mathscr{P} \in \mathscr{D}(X \times Y)$ , we can associate an exact functor of triangulated categories $\Phi_{\mathscr{P}} : \mathscr{D}(X) \longrightarrow \mathscr{D}(Y)$, which is defined as follows:

$$\Phi_{\mathscr{P}} : \mathscr{D}(X) \longrightarrow \mathscr{D}(Y)$$
$$\mathscr{F}^\bullet \mapsto Rp_*(\mathscr{P} \overset{L}{\otimes} Lq^*\mathscr{F}^\bullet).$$

**Notation 3.1.1.** Now and later, we will write $f_*$, $f^*$,$\otimes$, $\mathscr{H}om$, Hom respectively for $Lf_*$, $Rf^*$, $\overset{L}{\otimes}$, $R\mathscr{H}om$, $R$Hom: there is no risk of confusion, as we will always work in the derived context.

*Remark* 3.1.2. Be careful!The two notations $\mathrm{Hom}_{\mathscr{D}(X)}(\bullet,\bullet)$ and $\mathrm{Hom}(\bullet,\bullet)$ refer to different objects.

Let us notice some a basic consequence of the definition. The functor $\Phi_{\mathscr{P}}$ is called *Fourier-Mukai Transform of kernel $\mathscr{P}$.* Notice that it is always exact, because it is the composition of three exact functors, namely $p_*$, $q^*$ and $\mathscr{P}\otimes$, which we have proved to be exact in the first chapter. The exactness of a functor in the triangulated context is really important, much more than in the abelian context: an exact functor of triangulated categories commutes with the shift, which means more or less that we can treat each complex in the derived category like a direct sum of sheaves (in Proposition 3.1.7 we will prove that any complex is isomorphic to a direct sum of shifted sheaf). A triangulated functor which is not exact is, therefore, much less manageable: for this reason, we will focus our attention exclusively on exact functors all through the chapter.
*Properties:*

1. The identity

$$id : \mathscr{D}(X) \longrightarrow \mathscr{D}(X)$$

is isomorphic to the Fourier-Mukai transform with kernel $\mathscr{O}_\Delta$, where $\Delta \subset X \times X$ is the diagonal . Indeed, if we consider the diagonal embedding $i : X \xrightarrow{\cong} \Delta \subset X \times X$, then $i_*\mathscr{O}_X \cong \mathscr{O}_\Delta$, and for each $\mathscr{F}^\bullet \in \mathscr{D}(X)$ one has:

$$\begin{aligned}\Phi_{\mathscr{O}_\Delta}(\mathscr{F}^\bullet) &= p_*(\mathscr{O}_\Delta \otimes q^*\mathscr{F}^\bullet)\\ &\cong p_*(i_*\mathscr{O}_X \otimes q^*\mathscr{F}^\bullet)\\ &\cong p_*i_*(\mathscr{O}_X \otimes i^*q^*\mathscr{F}^\bullet)\\ &\cong (p\circ i)_*(\mathscr{O}_X \otimes (q\circ i)^*\mathscr{F}^\bullet)\\ &\cong \mathscr{F}^\bullet\end{aligned}$$

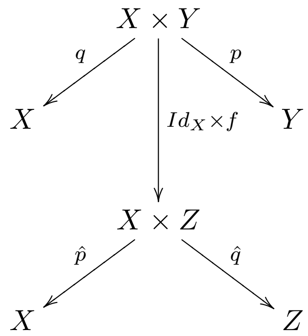
because $p \circ i = q \circ i = Id$.

2. Let $\mathscr{P} \in \mathscr{D}(X \times Y)$. Then:

- For each $f : Y \longrightarrow Z$:

$$f_* \circ \Phi_{\mathscr{P}} \cong \Phi_{(Id_X \times f)_* \mathscr{P}}$$

where $(Id_X \times f)_* \mathscr{P} \in \mathscr{D}(X \times Z)$. Indeed, consider the projections:



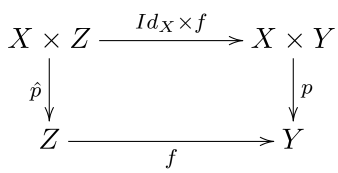

then one has:

$$\begin{aligned}
\Phi_{(Id_X \times f)_* \mathscr{P}}(\mathscr{F}^\bullet) &= \hat{p}_*((Id_X \times f)_* \mathscr{P} \otimes \hat{q}^* \mathscr{F}^\bullet) \\
&\underset{projection\ formula}{\cong} \hat{p}_*(Id_X \times f)_*(\mathscr{P} \otimes (Id_X \times f)^* \hat{q}^* \mathscr{F}^\bullet) \\
&\cong (\underbrace{\hat{p} \circ (Id_X \times f)}_{f \circ p})_*(\mathscr{P} \otimes (\underbrace{\hat{q} \circ (Id_X \times f)}_{q})^* \mathscr{F}^\bullet) \\
&\cong f_* p_*(q^* \mathscr{P} \otimes \mathscr{F}^\bullet) \\
&\cong f_* \circ \Phi_{\mathscr{P}}(\mathscr{F}^\bullet).
\end{aligned}$$

- For $f : Z \longrightarrow Y$:

$$f^* \circ \Phi_{\mathscr{P}} \cong \Phi_{(Id_X \times f)^* \mathscr{P}}$$

where $(Id_X \times f)^* \mathscr{P} \in \mathscr{D}(X \times Z)$. Indeed, if one considers the pullback square:



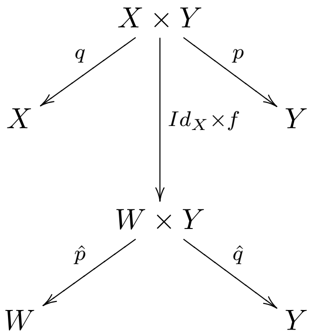

the flat base change gives

$$f^* p_* = \hat{p}_*(Id_X \times f)^*.$$

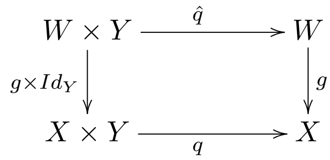
Therefore, an explicit computation gives:

$$
\begin{aligned}
f^* \circ \Phi_{\mathscr{P}}(\mathscr{F}^\bullet) &= f^* p_*(\mathscr{P} \otimes q^* \mathscr{F}^\bullet) \\
&\cong \hat{p}_* \circ (Id_X \times f)^* (\mathscr{P} \otimes q^* \mathscr{F}^\bullet) \\
&\cong \hat{p}_*((Id_X \times f)^* \mathscr{P} \otimes (Id_X \times f)^* q^* \mathscr{F}^\bullet) \\
&\cong \hat{p}_*((Id_X \times f)^* \mathscr{P} \otimes (\underbrace{q \circ (Id_X \times f)}_{\hat{q}})^* \mathscr{F}^\bullet) \\
&\cong \Phi_{(id_X \times f)^* \mathscr{P}}(\mathscr{F}^\bullet).
\end{aligned}
$$

- For $g : W \longrightarrow X$, the composition $\Phi_{\mathscr{P}} \circ g_*$ is isomorphic to the Fourier-Mukai transform $\Phi_{(g \times Id_Y)^* \mathscr{P}}$ with kernel $(g \times Id_Y)^* \mathscr{P} \in \mathscr{D}(W \times Y)$. If one calls the projections




then the flat base change on the pullback square




gives again:

$$q^* g_* = (g \times Id_Y)_* \hat{q}^*.$$

Therefore:

$$
\begin{aligned}
\Phi_{\mathscr{P}} \circ g_*(\mathscr{F}^\bullet) &= p_*(\mathscr{P} \otimes q^* g_* \mathscr{F}^\bullet) \\
&\underset{flat\ base\ change}{\cong} p_*(\mathscr{P} \otimes (g \times Id_Y)_* \hat{q}^* \mathscr{F}^\bullet) \\
&\underset{projection\ formula}{\cong} p_*(g \times Id_Y)_*((g \times Id_Y^* \mathscr{P} \otimes \hat{q}^* \mathscr{F}^\bullet) \\
&= (\underbrace{p \circ (g \times Id_Y)}_{\hat{p}})_*((g \times Id_Y^* \mathscr{P} \otimes \hat{q}^* \mathscr{F}^\bullet) \\
&= \Phi_{(g \times Id_Y)^* \mathscr{P}}(\mathscr{F}^\bullet).
\end{aligned}
$$

- For $g : X \longrightarrow W$ the composition $\Phi_{\mathscr{P}} \circ g^*$ is isomorphic to the Fourier-Mukai transform $\Phi_{(g \times Id_Y)_* \mathscr{P}}$. Indeed, just applying the projection formula, one gets:

$$\begin{aligned}\Phi_{(g\times Id_Y)_*\mathscr{P}}(\mathscr{F}^\bullet) &= \hat{p}_*((g \times Id_Y)_*\mathscr{P} \times \hat{q}\mathscr{F}^\bullet) \cong \\ &\cong \hat{p}_*(g \times Id_Y)_*(\mathscr{P} \otimes (g \times Id_Y))^*\hat{q}^*\mathscr{F}^\bullet) = \\ &= (\underbrace{\hat{p} \circ (g \times Id_Y)}_{p})_*(\mathscr{P} \otimes (\underbrace{\hat{q} \circ (g \times Id_Y)}_{g \circ q})^*\mathscr{F}^\bullet) = \\ &= p_*(\mathscr{P} \times q^* g^* \mathscr{F}^\bullet) = \\ &= \Phi_{\mathscr{P}} \circ g^*(\mathscr{F}^\bullet).\end{aligned}$$

3. If $f : X \longrightarrow Y$ is a morphism between algebraic varieties, then

$$f_* \cong \Phi_{\mathscr{O}_{\Gamma_f}} : \mathscr{D}(X) \longrightarrow \mathscr{D}(Y).$$

   Indeed, by an easy computation:

4. If $\mathscr{L}$ is a line bundle on $X$, then:

$$\Phi_{i_*\mathscr{L}} \cong \mathscr{L} \otimes (\bullet)$$

   where $i : X \xrightarrow{\cong} \Delta \subset X \times X$ is again the diagonal embedding. Indeed, the projection formula gives:

$$\begin{aligned}\Phi_{i_*\mathscr{L}}(\mathscr{F}^\bullet) &= p_*(i_*\mathscr{L} \otimes q^*\mathscr{F}^\bullet) \\ &\cong p_* i_*(\mathscr{L} \otimes i^* q^* \mathscr{F}^\bullet) \\ &= \mathscr{L} \otimes \mathscr{F}^\bullet.\end{aligned}$$

5. The shift functor $[1] : \mathscr{D}(X) \longrightarrow \mathscr{D}(X)$ is isomorphic to the FMT with kernel $\mathscr{O}_\Delta[1]$. The proof is very similar to the one of Item (1), and it simply exploits the fact that any exact functor of triangulated categories commutes with the shift:

$$\begin{aligned}\Phi_{\mathscr{O}_\Delta[1]}(\mathscr{F}^\bullet) &= p_*(\mathscr{O}_\Delta[1] \otimes q^*\mathscr{F}^\bullet) \\ &\cong p_*(\mathscr{O}_\Delta \otimes q^*\mathscr{F}^\bullet)[1] \\ &\cong [1] \circ \Phi_{\mathscr{O}_\Delta}(\mathscr{F}^\bullet) \\ &\cong \mathscr{F}^\bullet[1].\end{aligned}$$

6. Let $i : X \xrightarrow{\cong} \Delta \subset X \times X$ be the diagonal embedding again. Then:

$$\Phi_{i_* \omega_X^k} \cong S_X^k[-k \cdot \dim X].$$

Where $S$ is the Serre functor, as in Section 3.1 Indeed, by Item (4) and again by the fact that any exact functor commutes with the shift, one easily gets:

$$\begin{aligned}
(S_X^k[-k \cdot \dim X])(\mathscr{F}^\bullet) &= S_X^k(\mathscr{F}^\bullet[-k \cdot \dim X]) \\
&= \mathscr{F}^\bullet[-k \cdot \dim X] \otimes (\omega_X[\dim X])^k \\
&= \mathscr{F}^\bullet[-k \cdot \dim X] \otimes \underbrace{\omega_X[\dim X] \otimes ... \otimes \omega_X[\dim X]}_{k} \\
&= \mathscr{F}^\bullet[-k \cdot \dim X] \otimes (\underbrace{\omega_X \otimes ... \otimes \omega_X}_{k})[k \cdot \dim X] \\
&= \mathscr{F}^\bullet[-k \cdot \dim X] \otimes \omega_X^k[k \cdot \dim X] \\
&\cong \mathscr{F}^\bullet \otimes \omega_X^k \\
&\cong \Phi_{i_* \omega_X^k}(\mathscr{F}^\bullet).
\end{aligned}$$

7. Affare sulle deformazioni che non ho capito

8. The composition of two arbitrary FMT is again a FMT. Let $X, Y, Z$ be smooth projective varieties over a field $k$, as in the introduction to the Chapter. Consider objects $\mathscr{P} \in \mathscr{D}(X \times Y)$, $\mathscr{Q} \in \mathscr{D}(Y \times Z)$. Then define an object $\mathscr{R}$ in $\mathscr{D}(X \times Z)$ by the formula:

$$\mathscr{R} := \pi_{XZ_*}(\pi_{XY}^* \mathscr{P} \otimes \pi_{YZ}^* \mathscr{Q}),$$

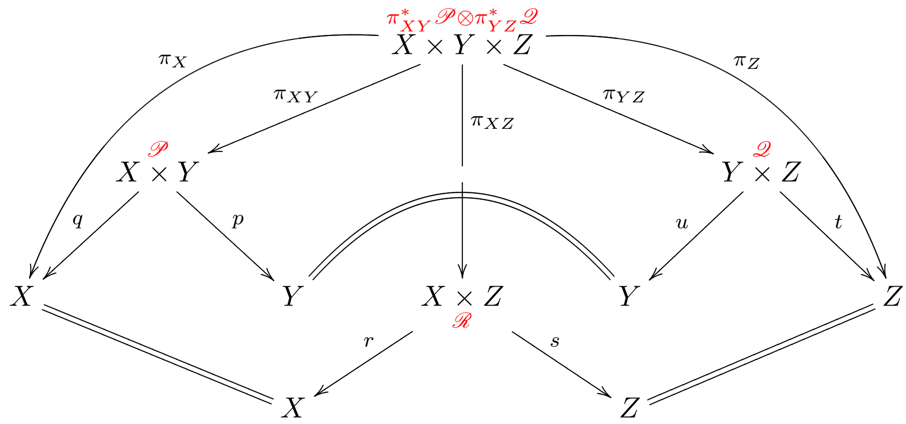
where

$$\begin{array}{ccccc} & & X \times Y \times Z & & \\ & \swarrow^{\pi_{XY}} & \downarrow{\scriptstyle \pi_{XZ}} & \searrow^{\pi_{YZ}} & \\ X \times Y & & X \times Z & & Y \times Z. \end{array}$$

Then one has $\Phi_{\mathscr{Q}} \circ \Phi_{\mathscr{P}} \cong \Phi_{\mathscr{R}}$, as displayed below:

$$\mathscr{D}(X) \xrightarrow{\Phi_{\mathscr{P}}} \mathscr{D}(Y) \xrightarrow{\Phi_{\mathscr{Q}}} \mathscr{D}(Z), \qquad \mathscr{D}(X) \xrightarrow{\Phi_{\mathscr{R}}} \mathscr{D}(Z).$$

For the proof, simply look at this commutative diagram:




then write down the following isomorphisms:

$$
\begin{aligned}
\Phi_{\mathscr{R}}(\mathscr{F}^\bullet) &= r_*(\mathscr{R}\otimes s^*\mathscr{F}^\bullet) \\
&\cong r_*(\pi_{XZ_*}(\pi_{XY}^*\mathscr{P}\otimes\pi_{YZ}^*\mathscr{Q})\otimes s^*\mathscr{F}^\bullet) \\
&\cong \underbrace{r_*\pi_{XZ_*}}_{(r\circ\pi_{XZ})_*=\pi_{Z_*}}(\pi_{XY}^*\mathscr{P}\otimes\pi_{YZ}^*\mathscr{Q}\otimes\underbrace{\pi_X^*}_{(q\circ\pi_{XY})^*}\mathscr{F}^\bullet) && \text{projection formula} \\
&\cong \underbrace{\pi_{Z_*}}_{(t\circ\pi_{YZ})_*}(\pi_{XY}^*(q^*\mathscr{F}^\bullet\otimes\mathscr{P})\otimes\pi_{YZ}^*\mathscr{Q}) \\
&\cong t_*\pi_{YZ_*}(\pi_{XY}^*(q^*\mathscr{F}^\bullet\otimes\mathscr{P})\otimes\pi_{YZ}^*\mathscr{Q}) \\
&\cong t_*(\pi_{XZ_*}\pi_{XY}^*(q^*\mathscr{F}^\bullet\otimes\mathscr{P})\otimes\mathscr{Q}) && \text{projection formula} \\
&\cong t_*(u^*p_*(q^*\mathscr{F}^\bullet\otimes\mathscr{P})\otimes\mathscr{Q}) && \pi_{YZ_*}\circ\pi_{XY}^*=u^*\circ p_* \\
&= t_*(u^*\Phi_{\mathscr{P}}(\mathscr{F}^\bullet)\otimes\mathscr{Q}) \\
&= \Phi_{\mathscr{Q}}(\Phi_{\mathscr{P}}(\mathscr{F}^\bullet)).
\end{aligned}
$$

where the passage $\pi_{YZ_*}\circ\pi_{XY}^*=u^*\circ p_*$ is given by the flat base change applied to the diagram

$$
\begin{array}{ccc}
X\times Y\times Z & \xrightarrow{\pi_{XY}} & X\times Y \\
{\scriptstyle\pi_{YZ}}\downarrow & & \downarrow{\scriptstyle p} \\
Y\times Z & \xrightarrow{u} & Y\ .
\end{array}
$$

9. Each FMT has a left and a right adjoint, which are both FMT, whose kernels can be explicitly described.

**Definition 3.1.3.** For each $\mathscr{P}\in\mathscr{D}(X\times Y)$, we set

$$
\begin{aligned}
\mathscr{P}_L &:= \mathscr{P}^\vee\otimes p^*\omega_Y[\dim Y] \\
\mathscr{P}_R &:= \mathscr{P}^\vee\otimes q^*\omega_X[\dim X]\ ,
\end{aligned}
$$

Therefore

$$\begin{aligned}\Phi_{\mathscr{P}_L}(\mathscr{F}^\bullet) &= q_*(\mathscr{P}^\vee \otimes p^*\omega_Y[\dim Y] \otimes p^*\mathscr{F}^\bullet)\\ &= q_*(\mathscr{P}^\vee \otimes p^*(\mathscr{F}^\bullet \otimes \omega_Y[\dim Y]))\\ &= \Phi_{\mathscr{P}^\vee}(S_Y(\mathscr{F}^\bullet)) = (\Phi_{\mathscr{P}^\vee} \circ S_Y)(\mathscr{F}^\bullet)\end{aligned}$$

and

$$\begin{aligned}\Phi_{\mathscr{P}_R}(\mathscr{F}^\bullet) &= q_*(\mathscr{P}_R \otimes p^*\mathscr{F}^\bullet)\\ &= q_*(\mathscr{P}^\vee \otimes p^*\omega_X[\dim X] \otimes p^*\mathscr{F}^\bullet)\\ &= q_*(\mathscr{P}^\vee \otimes p^*\mathscr{F}^\bullet) \otimes \omega_X[\dim X] && \text{(projection formula)}\\ &= S_X \circ \Phi^\vee(\mathscr{F}^\bullet) && .\end{aligned}$$

This yelds:

**Proposition 3.1.4.** *(Mukai) Let $\mathscr{P} \in \mathscr{D}(X \times Y)$. Then*

$$\Phi_{\mathscr{P}_L} \vdash \Phi_{\mathscr{P}} \vdash \Phi_{\mathscr{P}_R}.$$

*Proof.* For each $\mathscr{E}^\bullet \in \mathscr{D}(X)$, $\mathscr{F}^\bullet \in \mathscr{D}(Y)$, we have:

$$\begin{aligned}&\mathrm{Hom}_{\mathscr{D}(X)}(\Phi_{\mathscr{P}_L}(\mathscr{F}^\bullet), \mathscr{E}^\bullet)\\ &\quad= \mathrm{Hom}_{\mathscr{D}(X)}(q_*(\mathscr{P}^\vee \otimes p^*\omega_Y[\dim Y] \otimes p^*\mathscr{F}^\bullet), \mathscr{E}^\bullet)\\ &\qquad\qquad\qquad\qquad\qquad\qquad \text{(definition of } \Phi_{\mathscr{P}_L}(\mathscr{F}^\bullet))\\ &\quad= \mathrm{Hom}_{\mathscr{D}(X\times Y)}(\mathscr{P}^\vee \otimes p^*\omega_Y[\dim Y] \otimes p^*\mathscr{F}^\bullet, q^*\mathscr{E}^\bullet \otimes p^*\omega_Y[\dim Y])\\ &\qquad\qquad\qquad\qquad\qquad\qquad \text{(Grothendieck - Verdier duality)}\\ &\quad= \mathrm{Hom}_{\mathscr{D}(X\times Y)}(\mathscr{P}^\vee \otimes p^*\mathscr{F}^\bullet, q^*\mathscr{E}^\bullet)\\ &\qquad\qquad\qquad\qquad\qquad\qquad (S_Y \text{is fully faithful ???})\\ &\quad\cong \mathrm{Hom}_{\mathscr{D}(X\times Y)}(p^*\mathscr{F}^\bullet, \mathscr{P} \otimes q^*\mathscr{E}^\bullet)\\ &\qquad\qquad\qquad\qquad\qquad\qquad \text{(properties of } \mathscr{P}^\vee)\\ &\quad\cong \mathrm{Hom}_{\mathscr{D}(Y)}(\mathscr{F}^\bullet, p_*(\mathscr{P} \otimes q^*\mathscr{E}^\bullet))\\ &\qquad\qquad\qquad\qquad\qquad\qquad (p^* \vdash p_*)\\ &\quad\cong \mathrm{Hom}_{\mathscr{D}(Y)}(\mathscr{F}^\bullet, \Phi_{\mathscr{P}}(\mathscr{E}^\bullet)).\end{aligned}$$

And it similarly goes for $\Phi_{\mathscr{P}_R}$:

$$\begin{aligned}
&\operatorname{Hom}_{\mathscr{D}(X)}(\mathscr{E}^\bullet, \Phi_{\mathscr{P}_R}(\mathscr{F}^\bullet)) \\
&\qquad = \operatorname{Hom}_{\mathscr{D}(X)}(\mathscr{E}^\bullet, q_*(\mathscr{P}_R \otimes p^*\mathscr{F}^\bullet)) \\
&\qquad = \operatorname{Hom}_{\mathscr{D}(X\times Y)}(q^*\mathscr{E}^\bullet, \mathscr{P}_R \otimes p^*\mathscr{F}^\bullet) \\
&\qquad = \operatorname{Hom}_{\mathscr{D}(X\times Y)}(\mathscr{P} \otimes q^*\mathscr{E}^\bullet, q^*\omega_X[\dim X] \otimes p^*\mathscr{F}^\bullet) \\
&\qquad = \operatorname{Hom}_{\mathscr{D}(Y)}(p_*(\mathscr{P} \otimes q^*\mathscr{E}^\bullet)) \\
&\qquad = \operatorname{Hom}_{\mathscr{D}(Y)}(\Phi_{\mathscr{P}}(\mathscr{E}^\bullet), \mathscr{F}^\bullet)
\end{aligned}$$

□

The importance of Fourier-Mukai transforms is shown in the following Theorem, which is due to Orlov.

**Theorem 3.1.5.** *Let $X$ and $Y$ be shooth projective varieties and let*

$$F : \mathscr{D}(X) \longrightarrow \mathscr{D}(Y)$$

*be a fully faithful functor. If $F$ admits left and tight adjoint functors, then there exists an object $\mathscr{P} \in \mathscr{D}(X \times Y)$ such that*

$$\Phi_{\mathscr{P}} \cong F.$$

*Proof.* The proof is highly non-trivial, so we will just give references. There are two accounts of it in literature: the original one due to Orlov in [?], and another one due to Kawamata in [11]. □

The following example gives an idea of how one can lose information while passing from the objects in the derived category of the product to the corresponding Fourier-Mukai transforms.

**Example 3.1.6.** Let $X$ be an elliptic curve, and consider $\mathscr{O}_\Delta \in \mathscr{D}(X \times X)$. Using Serre duality, one gets that:

$$\begin{aligned}
\operatorname{Ext}^2(\mathscr{O}_\Delta, \mathscr{O}_\Delta) &\cong \operatorname{Ext}^0(\mathscr{O}_\Delta, \mathscr{O}_\Delta \otimes \underbrace{\omega_{X\times X}}_{\cong \mathscr{O}_{X\times X}}) \\
&\cong \operatorname{Hom}_{\mathscr{O}_{X\times X}}(\mathscr{O}_\Delta, \mathscr{O}_\Delta) \\
&\cong \operatorname{Hom}_{\mathscr{O}_{X\times X}}(\mathscr{O}_{X\times X}, \mathscr{O}_{X\times X})_{|\Delta} \\
&\cong \mathbb{C}
\end{aligned}$$

therefore $\operatorname{ext}^2(\mathscr{O}_\Delta, \mathscr{O}_\Delta) = \hom(\mathscr{O}_\Delta, \mathscr{O}_\Delta[2]) = 1$, which implies that there exists a nontrivial morphism

$$\phi : \mathscr{O}_\Delta \longrightarrow \mathscr{O}_\Delta[2].$$

Obviously, the morphism $\phi$ induces a morphism between the two Fourier-Mukai transforms corresponding to the two objects:

$$\Phi_\phi : \Phi_{\mathscr{O}_\Delta} \longrightarrow \Phi_{\mathscr{O}\Delta[2]}.$$

But we already know that

$$\Phi_{\mathscr{O}_\Delta} \cong Id_{\mathscr{D}(X)}$$

and

$$\Phi_{\mathscr{O}_\Delta[2]} \cong [2],$$

therefore the morphism of functors $\Phi_\phi$ is simply the double shift: $\mathscr{F}^\bullet \mapsto \mathscr{F}^\bullet[2]$. We now want to show that $\Phi_\phi$ is zero even if $\phi$ is not. For a sheaf $\mathscr{F} \in \mathrm{Coh}(X)$ this is trivial, as $\mathrm{Ext}^2(\mathscr{F}, \mathscr{F}) = 0$, by the fact that $\dim_{\mathbb{C}} X = 1$. To conclude, one uses the following

**Lemma 3.1.7.** *Let $X$ be a smooth projective curve. Then any object in $\mathscr{D}(X)$ is isomorphic to a direct sum of shifted sheaves $\bigoplus_{i \in I} \mathscr{E}_i[i]$, where $\mathscr{E}_i \in \mathrm{Coh}(X)$ for each $i$.*

*Proof.* Let $\mathscr{E}^\bullet \in \mathscr{D}(X)$. The proof goes by induction on the length of $\mathscr{E}^\bullet$ as a complex, i.e. the minimum $k = j - i$ such that $H^n(\mathscr{E}^\bullet) = 0$ for each $n \notin \{i, ..., j\}$. If the length of $\mathscr{E}^\bullet$ is one, then $\mathscr{E}^\bullet$ is itself a shifted sheaf, thus there is nothing to prove. Suppose now that the length of $\mathscr{E}^\bullet$ is $k > 1$. Then we can easily find a distinct triangle:

$$H^i(\mathscr{E}^\bullet)[-i] \longrightarrow \mathscr{E}^\bullet \longrightarrow \mathscr{E}_1^\bullet \xrightarrow{+1} H^i(\mathscr{E}^\bullet)[1-i]$$

where, as above, $i = \min\{\ell \mid H^n(\mathscr{E}^\bullet) = 0 \ \forall\ n < \ell\}$, and $\mathscr{E}_1^\bullet$ is a complex with length at most $k-1$. The idea is quite simple: suppose that $\mathscr{E}^\bullet = \{... \longrightarrow \mathscr{E}^{\ell-1} \xrightarrow{d^{\ell-1}} \mathscr{E}^\ell \xrightarrow{\ell} \mathscr{E}^{\ell+1} \longrightarrow ...\}$. Then the sheaf $H^i(\mathscr{E}^\bullet)[-i]$ is isomorphic, in the derived category, to the complex $\{... \longrightarrow 0 \longrightarrow ...\mathrm{Coker} d^{i-1} \xrightarrow{d^{i-1}} \mathscr{E}^i \xrightarrow{d^i} \mathrm{Im} d^i \longrightarrow 0 \longrightarrow ...\}$, which naturally embeds in $\mathscr{E}^\bullet$:

$$\begin{array}{ccccccccccccc}
... & \longrightarrow & \mathrm{Coker} d^{i-1} & \xrightarrow{d^{i-1}} & \mathscr{E}^i & \xrightarrow{d^i} & \mathrm{Im} d^i & \xrightarrow{d^{i+1}} & 0 & \longrightarrow & 0 & \longrightarrow & ... \\
 & & \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle \iota} & & \downarrow & & \downarrow & & \\
... & \longrightarrow & \mathrm{Coker} d^{i-1} & \xrightarrow{d^{i-1}} & \mathscr{E}^i & \xrightarrow{d^i} & \mathscr{E}^{i+1} & \xrightarrow{d^{i+1}} & \mathscr{E}^{i+2} & \xrightarrow{d^{i+2}} & \mathscr{E}^{i+3} & \longrightarrow & ...
\end{array}$$

the first two diagrams obviously commute, and the third one also does because $\mathscr{E}^\bullet$ is a complex, which means that the kernel of $d^i$ contains the image of $d^{i-1}$. Now, the quotient complex is

$$
\begin{array}{ccccccccccccc}
\dots & \longrightarrow & \mathrm{Coker}d^{i-1} & \xrightarrow{d^{i-1}} & \mathscr{E}^i & \xrightarrow{d^i} & \mathrm{Im}d^i & \xrightarrow{d^{i+1}} & 0 & \longrightarrow & 0 & \longrightarrow & \dots \\
 & & \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle \iota} & & \downarrow & & \downarrow & & \\
\dots & \longrightarrow & \mathrm{Coker}d^{i-1} & \xrightarrow{d^{i-1}} & \mathscr{E}^i & \xrightarrow{d^i} & \mathscr{E}^{i+1} & \xrightarrow{d^{i+1}} & \mathscr{E}^{i+2} & \xrightarrow{d^{i+2}} & \mathscr{E}^{i+3} & \longrightarrow & \dots \\
 & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \\
\dots & \longrightarrow & 0 & \xrightarrow{d^{i-1}} & 0 & \xrightarrow{d^i} & \mathrm{Coker}d^i & \xrightarrow{d^{i+1}} & \mathscr{E}_1^{i+2} & \xrightarrow{d^{i+2}} & \mathscr{E}_1^{i+3} & \longrightarrow & \dots
\end{array}
$$

where the complex $\mathscr{E}_1^\bullet = \{\dots \longrightarrow 0 \longrightarrow \mathrm{Coker}d^i \longrightarrow \mathscr{E}_1^{i+1} \longrightarrow \dots\}$ has length $k-1$. Now, if this distinguished triangle splits, i.e. $\mathscr{E}^\bullet \cong H^i(\mathscr{E}^\bullet)[-i] \oplus \mathscr{E}_1^\bullet$, then the induction hypotesis allows us to conclude. Thus, by property 5 of the distinguished triangles, it suffices to prove that $\mathrm{Hom}(\mathscr{E}_1^\bullet, H^i(\mathscr{E}^\bullet)[1-i]) = 0$. Use the inductive hypotesis again and write

$$\mathscr{E}_1^\bullet \cong \bigoplus_{i>k} H^i(\mathscr{E}_1^\bullet)[-k].$$

Therefore:

$$
\begin{aligned}
\mathrm{Hom}(\mathscr{E}_1^\bullet, H^i(\mathscr{E}^\bullet)[1-i]) &\cong \mathrm{Hom}(\bigoplus_{i>k} H^i(\mathscr{E}_1^\bullet)[-k], H^i(\mathscr{E}^\bullet[1-i]) \\
&\cong \bigoplus_{i>k} \mathrm{Hom}(H^i(\mathscr{E}_1^\bullet)[-k], H^i(\mathscr{E}^\bullet[1-i]) \\
&\cong \bigoplus_{i>k} \mathrm{Ext}^{1+k-i}(H^k(\mathscr{E}_1^\bullet), H^i(\mathscr{E}^\bullet)) = 0
\end{aligned}
$$

because $\dim X = 1$ and $k > i$.

□

This allows us to conclude: now take $\mathscr{F}^\bullet \in \mathscr{D}(X)$. The lemma above tells us that $\mathscr{F}^\bullet \cong \bigoplus_{i\in I} \mathscr{F}_i[i]$, therefore

$$
\begin{aligned}
\mathrm{Hom}(\mathscr{F}^\bullet, \mathscr{F}^\bullet[2]) &\cong \mathrm{Hom}(\bigoplus_{i\in I} \mathscr{F}_i[i], \bigoplus_{i\in I} \mathscr{F}_i[i+2]) \\
&\cong \bigoplus_{i\in I} \bigoplus_{j\in I} \mathrm{Hom}(\mathscr{F}_i[i], \mathscr{F}_j[j])
\end{aligned}
$$

and the shift map sends each $\mathscr{F}_i[i] \mapsto \mathscr{F}_i[i+2]$ and, for what said above, is zero.

# Chapter 4

# Stability conditions on K3 surfaces

In this chapter, we are going to give examples of stability conditions on the derived category of a K3 surface: this will show how the machinery we have developed in the first part applies to a concrete triangulated category, i.e. the derived category of a variety. We will not be dealing with abstract triangulated categories anymore: our objects of study, from now on, will be concrete derived categories, namely $\mathscr{D}(X) := \mathscr{D}^b(\mathrm{Coh}(X))$, where $X$ is an algebraic variety or, more specifically, a K3 surface. Recall that the objects of $\mathscr{D}(X)$ are complexes of coherent sheaves on $X$, which we will try to treat almost as sheaves, using the fact that there is a natural embedding $\mathrm{Coh}(X) \hookrightarrow \mathscr{D}(X)$, which simply comes from the fact that the abelian category $\mathrm{Coh}(X)$ is the heart of a bounded t-structure on $\mathscr{D}(X)$. We shall thus need to extend some of the general sheaf theory to complexes of sheaves: first of all, we would like to give some sense to sheaf invariants even for complexes. Recall that the *Chern character* of an invertible sheaf is:

$$\mathrm{ch}(\mathscr{L}) = \exp(\mathrm{c}_1(\mathscr{L})) = \sum_{k=0}^{+\infty} \frac{\mathrm{c}_1(\mathscr{L})^k}{k!}.$$

Notice that this definition naturally extends to arbitrary coherent shaves: first, to the locally free ones, by simply invoking the Splitting Principle, then to coherent sheaves using the fact that any coherent sheaf admits a finite locally free resolution. Since exact sequences in the Grothendieck group always split, and since the Chern classes are invariant under isomorphisms of sheaves, then the Chern Character as a ring homomorphism from the Grothendieck group of a variety to its rational cohomology ring is well defined[1]. Recall that the Grothendieck group has a natural ring structure, given as follows:

$$[E] + [F] := [E \oplus F] \ ;$$
$$[E] \cdot [F] := [E \otimes F]$$

for each $[E], [F] \in K(X)$. Now, we wonder how to further extend the definition of Chern classes and Chern character to complexes of sheaves. The most natural

[1]Actually, when dealing with K3 surface, it suffices to work with the integral cohomology ring, as the intersection form induced by the natural cup product is even.

definitions are the following ones:

$$\mathrm{c}_k(\mathscr{F}^\bullet) = \sum_{i=-\infty}^{+\infty} (-1)^i \mathrm{c}_k(H^i(\mathscr{F}^\bullet))$$

and

$$\mathrm{ch}(\mathscr{F}^\bullet) = \sum_{i=-\infty}^{+\infty} (-1)^i \mathrm{ch}(H^i(\mathscr{F}^\bullet))$$

for each $\mathscr{F}^\bullet \in K(X)$.[2] Notice that these definitions simply use the fact that any sheaf complex in the derived category is completely identified by its cohomology, and that the sums are indeed finite, due to the fact that we are working on the bounded derived category.
We shall count upon the theory introduced in the previous chapters to develop the necessary techniques. Our purposes, in this chapter, will be:

1. To give some generalities of the theory in the case of K3 surfaces. In particular, we will need to use the notion of *mukai vector* of an object $\mathscr{F}^\bullet$ in the derived category $\mathscr{D}(X)$ and to explain how it comes up to be important in the classification of the objects of $\mathscr{D}(X)$: it defines an isometry between the Grothendieck group of $X$ and the integral cohomology ring, when both are endowed with suitable bilinear forms. Then, we will examine the group $\mathrm{Aut}\mathscr{D}(X)$ of the exact autoequivalences of $X$, showing that there is a map between $\mathrm{Aut}\mathscr{D}(X)$ and the group of *Hodge isometries* os the integral cohomology lattice. We will need the theory developed in the previous chapter, as Fourier-Mukai transforms are needed. We will look at the kernel of this map, in order to describe some elements belonging to it. The principal aim of this will be to give all the needed definitions in order to state Theorem 4.1.8 and Conjecture 4.1.9, which are some of the principal results of the whole theory. The idea behind conjecture 4.1.9 is the following one: in the case of elliptic curves, the space of stability condition is not only completely determined, but also simply connected: it comes out to be homeomorphic to the universal cover of a connected component of some linear group. In the case of K3 surface, one's aim would be to reproduce this result, even if in a weaker version: what we hope is that there is at least a connected component (the one preserved by the action of $\mathrm{Aut}\mathscr{D}(X)$, see Chapter 2) which is simply connected. Moreover, Conjecture 4.1.9 states that the map between the group $\mathrm{Aut}\mathscr{D}(X)$ and the group of Hodge isometries of the integral cohomology lattice is surjective, giving an explicit description of its kernel.

2. To build a class of stability conditions, using the techniques provided by tilting theory. The fact is that, when dealing with derived categories, one has a preferred t-structure (the trivial one). Starting from this, it is easy to find new hearts, and therefore the chance to build suitable central charges raises. To find a new stability condition, we will use a pair of real divisors, where one of the two is ample, to build a torsion pair inside the category $\mathrm{Coh}(X)$ and,

[2] We will always denote with $H^i(\mathscr{F}^\bullet)$ the cohomology of the complex $\mathscr{F}^\bullet = \{\longrightarrow \mathscr{F}^{i-1} \longrightarrow \mathscr{F}^i \longrightarrow \mathscr{F}^{i+1} \longrightarrow ...\}$, i.e. $H^i(\mathscr{F}^\bullet) = \mathrm{Ker}d^i/\mathrm{Im}d^{i-1}$.

therefore, a new heart in $\mathscr{D}(X)$ by right tilting. Then, we will find a couple of central charges, both of the two compatible with the so-found heart, and we will explicitely show the compatibility. A very important result to recall will be Proposition 2.3.5, which gives a link between stability conditions on triangulated categories and hearts of bounded t-structures. This one will be the higher-dimensional known examples, as in dimension three no examples of stability conditions have been found yet.

3. To give a local characterization of the map

$$\pi : \mathrm{Stab}(X) := \mathrm{Stab}_{\mathscr{N}}(X) \longrightarrow \mathscr{N}(X) \otimes \mathbb{C},$$

mapping a stability condition to its central charge, which is slightly different from the one found in theorem 2.4.8. Notice that all through the chapter, the space $\mathrm{Stab}(X)$ will denote the space $\mathrm{Stab}_{\mathscr{N}}(X)$ of the stability conditions which factorize via the numerical Grothendieck group $\mathscr{N}(X)$. What will be shown is that, when restricted to the preimage of a certain subset of $\mathscr{N}(X) \otimes \mathbb{C}$, it defines a topological covering. What will be more difficult is to find the suitable subset: we will use the results developed in the first part and the properties of spherical objects.

4. To describe the wall-and-chamber structure of a connected component of the complex manifold $\mathrm{Stab}(X)$. We will prove that there is a collection of codimension-one submanifolds, called walls, which mark a border between stability and non-stability for certain objects. Roughly speaking: if one fixes a stability condition which is far from the walls and deforms it, some of the objects which are stable with respect to it remain stable until the deformed stability condition crosses one of the walls. This phenomenon is best-known as "wall-crossing". In particular, we will describe the walls and prove how the stability of some objects changes from a chamber to another. This result is important because it was developed by Arcara and Bertram in [**?**]: what they have found is that there is a link even between the Moduli spaces parametrizing the stable objects in each chamber. More specifically, one of that moduli spaces is a Moduli space of sheaves in the "classical" sense, and there are birational maps, called *Mukai flops*, which connect each of the Moduli space to another one.

Let us now give some basic results we are going to use all through the chapter.

## 4.1 Foundational material

Let $X$ be an algebraic K3 surface over $\mathbb{C}$. From now on, we will be principally dealing with two objects, i.e. the Grothendieck group $K(X)$ of $X$, which we have already defined, and the integral cohomology ring $H^*(X, \mathbb{Z})$. If we want to look for an application going from one to the other, the most natural choice could be:

$$\mathrm{ch} : K(X) \longrightarrow H^*(X, \mathbb{Z})$$

Now we want to endow both $K(X)$ and $H^*(X,\mathbb{Z})$ with a bilinear form. On the Grothendieck Group $K(X)$, we consider the *Euler bilinear form* $\chi(\bullet,\bullet)$, defined as:

$$\chi([E],[F]) = \sum_{i=-\infty}^{+\infty} (-1)^i \dim_{\mathbb{C}} \mathrm{Hom}(E, F[i]) \ .$$

**Notation 4.1.1.** We will usually write simply $\chi(E,F)$ instead of $\chi([E],[F])$, when there is no need to stress the fact that $E$ and $F$ are classes in the Grothendieck group, and $\hom(E,F)$, $\mathrm{ext}^i(E,F)$ respectively for $\dim_{\mathbb{C}} \mathrm{Hom}(E,F)$ and $\dim_{\mathbb{C}} \mathrm{Ext}^i(E,F)$.

This works because of the following fact: the derived category of a smooth projective variety over the complex numbers is of finite type, i.e. for each pair of objects $E, F \in \mathscr{D}(X)$, the complex vector space $\bigoplus_i \mathrm{Hom}_X(E, F[i])$ is finite-dimensional.[3]. Now, applying Serre's duality theorem, it is easy to show that the left and right radicals $K(X)^\perp$ and ${}^\perp K(X)$ with respect to the form $\chi$ are the same. Indeed, recall that:

$$K(X)^\perp := \{E \in K(X) \mid \chi(E,F) = 0 \ \ \forall F \in K(X)\}$$
$${}^\perp K(X) := \{E \in K(X) \mid \chi(F,E) = 0 \ \ \forall F \in K(X)\}.$$

So we need to show that if $E$ is in ${}^\perp K(X)$, then for each $F \in K(X)$ one has $\chi(E,F) = 0$. But the fact that $E \in^\perp K(X)$ means that for each $F \in K(X)$:

$$0 = \chi(E,F) = \hom(E,F) - \mathrm{ext}^1(E,F) + \mathrm{ext}^2(E,F)$$

This shows that the Euler form descends to a nondegenerate bilinear form on the *numerical Grothendieck group*

$$\mathscr{N}(X) = K(X)/K(X)^\perp \ .$$

On the other hand, we can define a symmetric bilinear form on the integral cohomology ring. We can write

$$H^*(X,\mathbb{Z}) = H^0(X,\mathbb{Z}) \oplus H^2(X,\mathbb{Z}) \oplus H^4(X,\mathbb{Z}) \cong \mathbb{Z} \oplus \mathrm{NS}(X) \oplus \mathbb{Z}$$

and endow it with the *Mukai bilinear form*, given by

$$((r,\Delta,s),(r',\Delta',s')) = rs' + r's - \Delta \cdot \Delta'$$

where the product $\Delta \cdot \Delta'$ is the usual cup product of the cohomology ring. This form is clearly bilinear and symmetric, and the resulting lattice is even of signature (3,19) (see the previous chapter about generalities on K3 surfaces).
Now, we want an isometry:

$$? : K(X)_{\chi(\cdot,\cdot)} \longrightarrow H^*(X,\mathbb{Z})_{(\cdot,\cdot)}.$$

First we notice that the Chern character does not work. Indeed, take the structure sheaf $\mathscr{O}_X$. Obviously:

[3] The reason is that the Ext groups are cohomologies, by Serre duality, and hence finite-dimensional.

$$\chi(\mathscr{O}_X, \mathscr{O}_X) = \hom(\mathscr{O}_X, \mathscr{O}_X) - \mathrm{ext}^1(\mathscr{O}_X, \mathscr{O}_X) + \mathrm{ext}^2(\mathscr{O}_X, \mathscr{O}_X) = 1 + 1 = 2$$

but

$$(\mathrm{ch}(\mathscr{O}_X), \mathrm{ch}(\mathscr{O}_X)) = ((1,0,0),(1,0,0)) = 0.$$

We need therefore to create something which works better. Luckily, we have a strong tool which suggests us the right solution: recall that the Hirzebruck-Riemann-Roch theorem states that if $X$ is a smooth compact complex variety and $\mathscr{F} \in \mathrm{Coh}(X)$, then

$$\chi(X, \mathscr{F}) = \int_X \mathrm{ch}(\mathscr{F})\mathrm{Td}(X)$$

where $\mathrm{Td}(X)$ is the *Todd class* of $X$, namely

$$\mathrm{Td}(X) := \mathrm{Td}(T_X) = \prod \frac{1}{1 - e^{\gamma_i}}$$

where the $\gamma_i$'s are the *Chern roots* of $\mathscr{F}$. Now, by Serre duality, we know that:

$$\mathrm{Ext}^i(\mathscr{E}, \mathscr{F}) \cong H^{2-i}(X, \mathscr{E}^* \otimes \mathscr{F})$$

therefore:

$$\chi(\mathscr{E}, \mathscr{F}) = \sum_i (-1)^i \mathrm{ext}^i(E, F) = \sum_i h^{2-i}(X, \mathscr{E}^* \otimes \mathscr{F}) \cong \sum_j h^j(X, \mathscr{E}^* \otimes \mathscr{F}) = \chi(\mathscr{E}^* \otimes \mathscr{F})$$

Notice, moreover, that if we denote with $\star$ the usual involution on the even part of the cohomology ring, namely:

$$\begin{array}{rccc} \star : & H^*(X,\mathbb{Z}) & \longrightarrow & H^*(X,\mathbb{Z}) \\ & \| & & \| \\ & H^0(X,\mathbb{Z}) \oplus H^2(X,\mathbb{Z}) \oplus H^4(X,\mathbb{Z}) & & H^0(X,\mathbb{Z}) \oplus H^2(X,\mathbb{Z}) \oplus H^4(X,\mathbb{Z}) \\ & (r \;,\; \Delta \;,\; s) & \longrightarrow & (r \;,\; -\Delta \;,\; s) \end{array}$$

we can observe that:

- if $u$ and $v$ are vectors in the lattice, then $(u, v) = \int_X u^\star \cdot v$, i.e., the top degree part of the usual cup product of $u$ and $v$ . In fact, if $u = (r, \Delta, s)$ and $v = (r', \Delta', s')$ then:

$$\begin{aligned}\int_X u^\star \cdot v &= (r, -\Delta, s)\cdot(r', \Delta', s')\\ &= [rr' + r\Delta' + rs' - \Delta r' - \Delta\Delta' - \Delta s' + sr' + ss']_4\\ &= rs' - \Delta\Delta' + sr'\\ &= (u, v).\end{aligned}$$

- If $\mathscr{E}$ is a locally free sheaf of rank $n$, then $\mathrm{ch}(\mathscr{E})^\star = \mathrm{ch}(\mathscr{E}^*)$ where $\mathscr{E}^* = \mathrm{Hom}(\mathscr{E}, \mathscr{O}_X)$. Actually, we immediately have that $\mathrm{rk}(\mathscr{E}) = \mathrm{rk}(\mathscr{E}^*)$, because the rank of $\mathscr{E}$ as a locally free sheaf coincides with the rank of the corresponding vector bundle; therefore:

$$\mathrm{rk}(\mathscr{E}^*) = \dim_{\mathbb{C}}(\mathscr{E})^*(x) = \dim_{\mathbb{C}}(\mathscr{E}(x))^* = \dim_{\mathbb{C}}(\mathscr{E}(x)) = \mathrm{rk}(\mathscr{E})$$

where $x \in X$ and $\mathscr{E}(x)$ is the *fiber* of $\mathscr{E}$ at x, namely $\mathscr{E}(x) = \mathscr{E}_x/\mathscr{M}_x\mathscr{E}_x$ and $\mathscr{M}_x$ is the maximal ideal of $\mathscr{O}_{X,x}$. Moreover, using the fact that the Chern character is a group homomorphism:

$$\mathrm{ch}(\mathscr{E}^*)\mathrm{ch}(\mathscr{E}) = \mathrm{ch}(\mathscr{E}^* \otimes \mathscr{E}) = \mathrm{ch}(\mathscr{O}_X^{n^2}),$$

we get that:

$$\begin{aligned}&(\mathrm{rk}(\mathscr{E}^*), c_1(\mathscr{E}^*), \frac{1}{2}c_1(\mathscr{E}^*)^2 - c_2(\mathscr{E}^*))\cdot(\mathrm{rk}(\mathscr{E}), c_1(\mathscr{E}), \underbrace{\frac{1}{2}c_1(\mathscr{E})^2 - c_2(\mathscr{E})}_{\mathrm{ch}_2(\mathscr{E})}) =\\ &= (\mathrm{rk}(\mathscr{E}^*)\mathrm{rk}(\mathscr{E}), \mathrm{rk}(\mathscr{E}^*)c_1(\mathscr{E}) + \mathrm{rk}(\mathscr{E})c_1(\mathscr{E}^*), \mathrm{rk}(\mathscr{E})\mathrm{ch}_2(\mathscr{E}^*) + c_1(\mathscr{E})c_1(\mathscr{E}^*) + rk(\mathscr{E}^*)\mathrm{ch}_2(\mathscr{E}))\\ &= (n^2, 0, 0)\end{aligned}$$

therefore, from the fact that $\mathrm{rk}(\mathscr{E}) = \mathrm{rk}(\mathscr{E}^*)$ we get:

$$\mathrm{rk}(\mathscr{E}^*)c_1(\mathscr{E}) + \mathrm{rk}(\mathscr{E})c_1(\mathscr{E}^*) = \mathrm{rk}(\mathscr{E})c_1(\mathscr{E}) + \mathrm{rk}(\mathscr{E})c_1(\mathscr{E}^*) = 0$$

which implies that $c_1(\mathscr{E}^*) = -c_1(\mathscr{E})$. Finally, the degree three - part of the product, which must be zero for dimensional reasons, gives that:

$$\begin{aligned}c_1(\mathscr{E}^*)\mathrm{ch}_2(\mathscr{E}) + &c_1(\mathscr{E})\mathrm{ch}_2(\mathscr{E}) =\\ &= c_1(\mathscr{E}^*)(\frac{1}{2}c_1(\mathscr{E})^2 - c_2(\mathscr{E})) + c_1(\mathscr{E})\frac{1}{2}c_1(\mathscr{E}^*)^2 - c_2(\mathscr{E}^*) =\\ &= \frac{1}{2}c_1(\mathscr{E})^3 - \frac{1}{2}c_1(\mathscr{E})^3 - c_1(\mathscr{E})c_2(\mathscr{E}^*) + c_1(\mathscr{E})c_2(\mathscr{E}) = 0\end{aligned}$$

and $c_2(\mathscr{E}^*) = c_2(\mathscr{E})$. This means that $\mathrm{ch}(\mathscr{E}^*) = (\mathrm{rk}(\mathscr{E}), -c_1(\mathscr{E}^*), \frac{1}{2}c_1(\mathscr{E})^2 - c_2(\mathscr{E})) = \mathrm{ch}(\mathscr{E})^\star$.

**Proposition 4.1.2.** *The Mukai vector*

$$v : K(X)_{\chi(\cdot,\cdot)} \longrightarrow H^*(X,\mathbb{Z})_{(\cdot,\cdot)}.$$

*is an isometry.*

*Proof.* Using what we discovered in the previous remarks:

$$\begin{aligned}
(v(\mathscr{E}), v(\mathscr{F})) &= \int_X v(\mathscr{E})^\star v(\mathscr{F}) \\
&= \int_X v(\mathscr{E}^*) v(\mathscr{F}) \\
&= \int_X \mathrm{ch}(\mathscr{E}^*)\sqrt{\mathrm{Td}(X)}\mathrm{ch}(\mathscr{F})\sqrt{\mathrm{Td}(X)} \\
&= \int_X \mathrm{ch}(\mathscr{E}^*)\mathrm{ch}(\mathscr{F})\mathrm{Td}(X) \\
&= \int_X \mathrm{ch}(\mathscr{E}^* \otimes \mathscr{F})\mathrm{Td}(X) \\
&\overset{HRR}{=} \chi(X, \mathscr{E}^* \otimes \mathscr{F}) \\
&= \chi(\mathscr{E}, \mathscr{F})
\end{aligned}$$

□

**Proposition 4.1.3.** *The Mukai vector identifies the numerical Grothendieck group $\mathscr{N}(X)$ with the cohomology sublattice $\mathbb{Z} \oplus \mathrm{NS}(X) \oplus \mathbb{Z} \subset H^*(X,\mathbb{Z}) = H^0(X,\mathbb{Z}) \oplus H^2(X,\mathbb{Z}) \oplus H^4(X,\mathbb{Z})$*

*Proof.* We already know that the Mukai vector defines an isometry:

$$v : K(X) \longrightarrow H^*(X,\mathbb{Z}) ,$$

so what we need to prove is that

- it factors through the quotient $K(X)/K(X)^\perp$,
- it is injective
- its image is the lattice $\mathbb{Z} \oplus \mathrm{NS}(X) \oplus \mathbb{Z}$.

Let us prove the third point first, i.e., that for each $u \in \mathbb{Z} \otimes \mathrm{NS}(X) \otimes \mathbb{Z}$, there exists an object $\mathscr{E} \in K(X)$ such that $v(\mathscr{E}) = u$. Let $u = (r, \Delta, s)$. The class $\Delta$ belongs to $\mathrm{NS}(X)$, which is, by definition, the image of the first Chern class

$$c_1 : \mathrm{Pic}(X) \longrightarrow H^2(X,\mathbb{Z}) ,$$

therefore we will always find a line bundle $\mathscr{L}$ such that $c_1(\mathscr{L}) = \Delta$. Now, if $r > 0$, then the rank and the first Chern class of the sheaf $\mathscr{L} \oplus \mathscr{O}_X^{r-1}$ will be the required ones, i.e., the first two components of $u$. If $r < 0$, then we will need a two terms-complex, i.e. $\mathscr{E}^\bullet = \{... \longrightarrow \mathscr{E}^{-1} \longrightarrow \mathscr{E}^0 \longrightarrow \mathscr{E}^1 \longrightarrow \mathscr{E}^2 \longrightarrow ...$, such that

$$H^i(\mathscr{E}^\bullet) = \begin{cases} \mathscr{L} & \text{if } i = 0, \\ \mathscr{O}_X^{r+1} & \text{if } i = 1, \\ 0 & \text{otherwise} \end{cases} .$$

Indeed, recall that the rank and the Chern classes of a complex are the alternating sums of the ranks and the Chern classes of its cohomologies. Finally, given the third component of the Mukai vector, one can build a sheaf which gives no contribution to the rank and to the first Chern class, but whose second Chern class is the required one. Indeed, consider a sheaf $\mathscr{F}$ supported on a curve $C$ contained in $X$, and let $i : C \hookrightarrow X$ be the immersion. Then $i_*\mathscr{F}$ is a sheaf on $X$ with both rank and first Chern class zero. Then, again by Hirzebruch-Riemann-Roch, we have:

$$\chi(X, i_*\mathscr{F}) = \int_X \mathrm{ch}(\mathscr{F})\mathrm{Td}(X) = \int_X -c_2(\mathscr{F}) \cdot (1 + \frac{c_2(X)}{12}) = -c_2(\mathscr{F})$$

but on the other hand $\chi(X, i_*\mathscr{F}) = \chi(C, \mathscr{F})$ which is equal, by Riemann-Roch for curves, to $d - g + 1$, where $d = \deg(\mathscr{F})$ and $g$ is the genus of $C$. Therefore, being $d$ and $g$ integers which can vary freely, we can produce a sheaf with fixed second Chern class. Setting now $\mathscr{G}^\bullet := \mathscr{E}^\bullet \oplus \mathscr{F}$, we have found the required object.
Now let us turn our attention to the other two points. It suffices to prove that:

1. $K(X)^\perp$ is a subgroup of $K(X)$ (it is obviously normal, because $K(X)$ is abelian);

2. $K(X)^\perp = \mathrm{Ker} v$, where $v$ is seen as a homomorphisms of abelian groups.

If 1) and 2) hold, then we can apply the first Theorem of homomorphism and conclude that the map $v$ factors through the quotient group $\mathscr{N}(X) = K(X)/K(X)^\perp$.

1. To show that $K(X)^\perp$ is a subgroup of $K(X)$, it suffices to prove that for each pair of objects $[\mathscr{E}^\bullet], [\mathscr{F}^\bullet] \in K(X)^\perp$, one has $[\mathscr{E}^\bullet] - [\mathscr{F}^\bullet] \in K(X)^\perp$[4]. But we know that $-[\mathscr{F}^\bullet] = [\mathscr{F}^\bullet[1]]$, therefore $[\mathscr{E}^\bullet] - [\mathscr{F}^\bullet] = [\mathscr{E}^\bullet \oplus \mathscr{F}^\bullet[1]]$. Thus for each $\mathscr{G}^\bullet \in K(X)$:

$$\begin{aligned} \chi([\mathscr{E}^\bullet - \mathscr{F}^\bullet], [\mathscr{G}^\bullet]) &= \chi([\mathscr{E}^\bullet \oplus \mathscr{F}^\bullet[1]], [\mathscr{G}^\bullet]) \\ &= \underbrace{\chi([\mathscr{E}^\bullet], [\mathscr{G}^\bullet])}_{\substack{\| \\ 0}} + \chi([\mathscr{E}^\bullet[1]], [\mathscr{G}^\bullet]) \\ &= -\chi([\mathscr{F}^\bullet], [\mathscr{G}^\bullet]) \\ &= 0 . \end{aligned}$$

[4]Here I am using square parentheses because I want to stress the fact that we are dealing with equivalence classes, and make a distinction between the classes and the objects representing them in the derived category

2. What we need to use is just that, if $\mathscr{E} \in K(X)$, then $v(\mathscr{E}) = (0,0,0)$ if and only if $(v(\mathscr{E}), u) = 0$ for each $u \in \mathbb{Z} \oplus \mathrm{NS}(X) \oplus \mathbb{Z}$. If $v(\mathscr{E}) = (0,0,0)$, one obviously has $(v(\mathscr{E}), u) = 0$ for each $u \in \mathbb{Z} \oplus \mathrm{NS}(X) \oplus \mathbb{Z}$. For the converse, consider $v(\mathscr{E}) = (r, \Delta, s)$ such that $rs' + r's - \Delta \cdot \Delta' = 0$ for each $(r', \Delta', s') \in \mathbb{Z} \oplus \mathrm{NS}(X) \oplus \mathbb{Z}$. Suppose that $(r, \Delta, s) \neq (0,0,0)$, and that, for example, $r \neq 0$. But if $(r', \Delta', s') = (0, 0, s')$ with $s' \neq 0$, then

$$((r, \Delta, s), (r', \Delta', s')) = rs' \neq 0$$

which is a contradiction. Same goes for the other two components. But this means that $\mathscr{E} \in \mathrm{Ker} v \Leftrightarrow v(\mathscr{E}) = (0,0,0) \Leftrightarrow (v(\mathscr{E}), u) = 0$ for each $u \in \mathbb{Z} \oplus \mathrm{NS}(X) \oplus \mathbb{Z} \Leftrightarrow (v(\mathscr{E}), v(\mathscr{F})) = 0$ for each $\mathscr{F} \in K(X) \Leftrightarrow \chi(\mathscr{E}, \mathscr{F}) = 0$ for each $\mathscr{F} \in K(X) \Leftrightarrow \mathscr{E} \in K(X)^{\perp}$.

□

We are now ready to give the basic ingredients of the main theorem of this section.

**Definition 4.1.4.** An isometry $f : H^*(X, \mathbb{Z}) \longrightarrow H^*(X, \mathbb{Z})$ is called a *Hodge isometry* if it is a morphism of Hodge structures, i.e. $f(H^{p,q}(X, \mathbb{Z})) = H^{p,q}(X, \mathbb{Z})$ for all $p, q$. The group of Hodge isometries of a K3 surface X is denoted with $\mathrm{Aut}H^*(X, \mathbb{Z})$.

A classical result by Orlov states that every exact autoequivalence of $\mathscr{D}(X)$ induces a Hodge isometry, i.e. there is a map

$$\omega : \mathrm{Aut}\mathscr{D}(X) \longrightarrow \mathrm{Aut}H^*(X, \mathbb{Z})$$

whose kernel is denoted $\mathrm{Aut}^0\mathscr{D}(X)$. Let us briefly outline how it comes out.
Consider an exact autoequivalence $\Phi \in \mathrm{Aut}\mathscr{D}(X)$. Since an autoequivalence always admits left and right adjoint, which both coincide with its inverse, Theorem 3.1.5 assures us that there exists an object $\mathscr{E}^\bullet \in \mathscr{D}(X \times X)$ such that $\Phi \cong \Phi_{\mathscr{E}^\bullet}$. Now, consider the algebraic cycle:

$$Z := q^* \sqrt{\mathrm{Td}(X)} \cdot \mathrm{ch}(\mathscr{E}^\bullet) \cdot p^* \sqrt{\mathrm{Td}(X)}.$$

This cycle defines an endomorphism of integral cohomology of $X$:

$$\begin{array}{cccc} f: & H^*(X,\mathbb{Z}) & \longrightarrow & H^*(X,\mathbb{Z}) \\ & \cup & & \cup \\ & v & \mapsto & p_*(Z \cdot q^*(v)) \end{array}$$

Therefore we have the following:[5]

[5]The result also holds in a more general case, i.e. when $\Phi_{\mathscr{E}^\bullet}$ is fully faithful, by a classical result which shows that a fully faithful functor always admits left and right adjoints. However, for our purposes it suffices to consider only the case of autoequivalences

**Proposition 4.1.5.** *If* $\Phi_{\mathscr{E}^\bullet} \in \mathrm{Aut}\mathscr{D}(X)$, *then:*

1. *$f$ is an isometry from the lattice associated to $H^*(X,\mathbb{Z})$ to itself;*
2. *the inverse of $f$ is equal to the homomorphism*

$$\begin{array}{cccc} \hat{f}: & H^*(X,\mathbb{Z}) & \longrightarrow & H^*(X,\mathbb{Z}) \\ & \cup & & \cup \\ & v & \mapsto & q_*(Z^\vee \cdot p^*(v)) \end{array}$$

*where* $Z^\vee := q^*\sqrt{\mathrm{Td}(X)} \cdot \mathrm{ch}((\mathscr{E}^\bullet)^\vee) \cdot p^*\sqrt{\mathrm{Td}(X)}$ .

*Proof.* Recall that the left and right adjoints of $\Phi_{\mathscr{E}^\bullet}$ are

$$\begin{aligned} \Phi_{\mathscr{E}^\bullet_R} &= q_*((\mathscr{E}^\bullet)^\vee \otimes p^*(\bullet)) \otimes \omega_X[\dim X] \\ &= q_*((\mathscr{E}^\bullet)^\vee \otimes p^*(\bullet)) \otimes \mathscr{O}_X[2] \\ &= q_*((\mathscr{E}^\bullet)^\vee \otimes p^*(\bullet))[2] \\ &= \Phi_{\mathscr{E}^\bullet_L}. \end{aligned}$$

Since $\Phi_{\mathscr{E}^\bullet}$ is an autoequivalence, the compositions $\Phi_{\mathscr{E}^\bullet_L} \circ \Phi_{\mathscr{E}^\bullet} \cong \Phi_{\mathscr{E}^\bullet} \circ \Phi_{\mathscr{E}^\bullet_R}$ are isomorphic to the identity $Id_{\mathscr{D}(X)} \cong \Phi_{\mathscr{O}_\Delta}$, with $\Delta \subset X \times X$ is the diagonal, as usual. Now, using the projection formula and Grothendieck-Riemann-Roch theorem, we get:

$$\hat{f} \circ f = Id_{H^*(X,\mathbb{Z})}$$

Therefore, $\hat{f} \circ f$ is the identity and hence $f$ is an automorphism of the lattice. Now, taking $u$ and $v$ in $H^*(X,\mathbb{Z})$, a little bit of computation gives:

$$\begin{aligned} (v, f(u)) &= \int_X v^\vee \cdot p_*(p^*\sqrt{\mathrm{Td}(X)} \cdot \mathrm{ch}(\mathscr{E}^\bullet) \cdot q^*\sqrt{\mathrm{Td}(X)} \cdot q^*(u)) \\ &= \int_X p_*(p^*(v^\vee) \cdot q^*(u) \cdot \mathrm{ch}(\mathscr{E}^\bullet) \cdot \sqrt{\mathrm{Td}(X \times X)}) && \text{(projection formula)} \\ &= \int_{X \times X} p^*(v^\vee) \cdot q^*(u) \cdot \mathrm{ch}(\mathscr{E}^\bullet) \cdot \sqrt{\mathrm{Td}(X \times X)} \end{aligned}$$

and similarly

$$(u, \hat{f}(v)) = \int_{X \times X} (q^*(u^\vee) \cdot p^*(v) \cdot \mathrm{ch}(\mathscr{E}^\bullet)^\star \cdot \sqrt{\mathrm{Td}(X \times X)},$$

hence $(v, f(u)) = (\hat{f}(v), u)$. Since $\hat{f} \circ f$ is the identity, we get:

$$(f(u), f(v)) = (\hat{f}(f(u)), v) = (u, v).$$

Thus, $f$ is an isometry.

□

Three interesting classes of exact autoequivalences are:

1. the shift functor $[1] : \mathscr{D}(X) \longrightarrow \mathscr{D}(X)$;

2. pullbacks by automorphisms of $X$: if $\phi : X \xrightarrow{\cong} X$ is an automorphism, then $\phi^* : \mathrm{Coh}(X) \longrightarrow \mathrm{Coh}(X)$ defines an autoequivalence of abelian categories which can be extended to an exact autoequivalence of derived categories, $\hat{\phi}^* : \mathscr{D}(X) \longrightarrow \mathscr{D}(X)$;

3. Twists by spherical objects.

Recall that an object $\mathscr{E}^\bullet \in \mathscr{D}(X)$, where $X$ is an arbitrary algebraic complex variety, is called *spherical* if:

- $\mathscr{E}^\bullet \otimes \omega_X \cong \mathscr{E}^\bullet$;
- $\mathrm{Ext}^i(\mathscr{E}^\bullet, \mathscr{E}^\bullet) = \begin{cases} \mathbb{C} & \text{if } i = 0, \dim X \\ 0 & \text{otherwise} \end{cases}$

To each complex of locally free sheaves $\mathscr{E}^\bullet \in \mathscr{D}(X)$ we can associate a functor which is called *twist functor* $T_{\mathscr{E}^\bullet}$, defined as the Fourier-Mukai transform with kernel

$$\mathscr{P} = \mathrm{Cone}(\eta : (\mathscr{E}^\bullet)^\vee \boxtimes \mathscr{E}^\bullet \longrightarrow \mathscr{O}_\Delta)$$

where $\Delta \subset X \times X$ is the diagonal, and the map $\eta$ is the canonical pairing. Since quasi-isomorphic complexes give rise to isomorphic Fourier-Mukai transforms, one can use the fact that each coherent sheaf has a finite locally-free resolution to extend the definition to each object $\mathscr{E}^\bullet \in \mathscr{D}(X)$. If $\mathscr{E}^\bullet$ is spherical, moreover, a result shows that the corresponding twist functor is an exact autoequivalence of $\mathscr{D}(X)$.
Let us call

$$T_{\mathscr{E}^\bullet} := \Phi_{\mathscr{P}}$$

Then we want to give a more explicit characterization of $T_{\mathscr{E}^\bullet}(\mathscr{F})$. Consider an arbitrary distinguished triangle:

$$\mathscr{E}^\bullet \longrightarrow \mathscr{F}^\bullet \longrightarrow \mathscr{G}^\bullet \xrightarrow{+1}$$

Where $\mathscr{E}^\bullet, \mathscr{F}^\bullet$ and $\mathscr{G}^\bullet$ are in $\mathscr{D}(X \times X)$. Then, if $q$ and $p$ are the two projections respectively on the first and on the second factor and $\mathscr{H}^\bullet \in \mathscr{D}(X)$, both

$$\mathscr{E}^\bullet \otimes q^*\mathscr{H}^\bullet \longrightarrow \mathscr{F}^\bullet \otimes q^*\mathscr{H}^\bullet \longrightarrow \mathscr{G}^\bullet \otimes q^*\mathscr{H}^\bullet \xrightarrow{+1}$$

$$p_*(\mathscr{E}^\bullet \otimes q^*\mathscr{H}^\bullet) \longrightarrow p_*(\mathscr{F}^\bullet \otimes q^*\mathscr{H}^\bullet) \longrightarrow p_*(\mathscr{G}^\bullet \otimes q^*\mathscr{H}^\bullet) \xrightarrow{+1}$$

are distinguished triangle, because all the functors involved are exact functors of triangulated categories, so we can conclude that

$$\Phi_{\mathscr{E}^\bullet}(\mathscr{H}^\bullet) \longrightarrow \Phi_{\mathscr{F}^\bullet}(\mathscr{H}^\bullet) \longrightarrow \Phi_{\mathscr{G}^\bullet}(\mathscr{H}^\bullet) \xrightarrow{+1}$$

is a distinguished, as well. More explicitely, the distinguished triangle we are interested in is:

$$(\mathscr{E}^\bullet)^\vee \boxtimes \mathscr{E}^\bullet \longrightarrow \mathscr{O}_\Delta \longrightarrow \mathrm{Cone}(\eta : (\mathscr{E}^\bullet)^\vee \boxtimes \mathscr{E}^\bullet \longrightarrow \mathscr{O}_\Delta) \xrightarrow{+1}$$

Where, as usual, $(\mathscr{E}^\bullet)^\vee := R\mathscr{H}om(\mathscr{E}^\bullet, \mathscr{O}_X)$. Therefore

$$\Phi_{(\mathscr{E}^\bullet)^\vee \boxtimes \mathscr{E}^\bullet}(\mathscr{F}^\bullet) \longrightarrow \Phi_{\mathscr{O}_\Delta}(\mathscr{F}^\bullet) \longrightarrow T_{\mathscr{E}^\bullet}(\mathscr{F}^\bullet) \xrightarrow{+1}$$

gives that $T_{\mathscr{E}^\bullet}(\mathscr{F}^\bullet)$ is a cone of some morphism. Now, by the general theory, we know that

$$\Phi_{\mathscr{O}_\Delta}(\mathscr{F}^\bullet) \cong \mathscr{F}^\bullet$$

and some computation gives that:

$$\begin{aligned}
\Phi_{(\mathscr{E}^\bullet)^\vee \boxtimes \mathscr{E}}(\mathscr{F}) =& p_*(q^*(\mathscr{E}^\bullet)^\vee \otimes p^*\mathscr{E}^\bullet \otimes q^*\mathscr{F}^\bullet) \\
&\cong p_*(q^*((\mathscr{E}^\bullet)^\vee \otimes \mathscr{F}^\bullet) \otimes p^*\mathscr{E}^\bullet) \\
&\cong p_*(q^*(\mathscr{E}^\bullet)^\vee \otimes \mathscr{F}^\bullet) \otimes \mathscr{E}^\bullet && \text{(projection formula)} \\
&\cong (\mathscr{E}^\bullet)^\vee \otimes \mathscr{F}^\bullet \otimes \mathscr{E}^\bullet && \text{(flat base change)} \\
&\cong \mathrm{Hom}^\bullet(\mathscr{E}^\bullet, \mathscr{F}^\bullet) \otimes \mathscr{E}^\bullet && .
\end{aligned}$$

Therefore, for each $\mathscr{F} \in \mathscr{D}(X)$, the twist $T_{\mathscr{E}}(\mathscr{F})$ of $\mathscr{F}$ by $\mathscr{E}$ fits into a distinguished triangle:

$$\mathrm{Hom}^\bullet(\mathscr{E}^\bullet, \mathscr{F}^\bullet) \otimes \mathscr{E}^\bullet \longrightarrow \mathscr{F}^\bullet \longrightarrow T_{\mathscr{E}^\bullet}(\mathscr{F}^\bullet)$$

$$T_{\mathscr{E}^\bullet}(\mathscr{F}^\bullet) = \mathrm{Cone}(\mathrm{Hom}^\bullet(\mathscr{E}^\bullet, \mathscr{F}^\bullet) \otimes \mathscr{E}^\bullet) \longrightarrow \mathscr{F}^\bullet),$$

where

$$\mathrm{Hom}^\bullet(\mathscr{E}^\bullet, \mathscr{F}^\bullet) \otimes \mathscr{E}^\bullet$$

stands for

$$\bigoplus_i \mathrm{Hom}^\bullet(\mathscr{E}^\bullet, \mathscr{F}[i]^\bullet) \otimes \mathscr{E}[-i]^\bullet.$$

Since $\mathscr{E}^\bullet$ is spherical, one has

$$\mathrm{Hom}^\bullet(\mathscr{E}^\bullet, \mathscr{E}[i]^\bullet) = \mathrm{Ext}^i(\mathscr{E}, \mathscr{E}) = \mathbb{C} \text{ if } i = 0, 2, \ 0 \text{ otherwise} .$$

Therefore

$$\bigoplus_i \mathrm{Hom}^\bullet(\mathscr{E}^\bullet, \mathscr{E}[i]^\bullet) \otimes \mathscr{E}[-i]^\bullet = \mathbb{C} \otimes \mathscr{E}^\bullet \oplus \mathbb{C} \otimes \mathscr{E}[-2]^\bullet = \mathscr{E}^\bullet \oplus \mathscr{E}[-2]^\bullet$$

So one looks for the cone of

$$\mathscr{E}^\bullet \oplus \mathscr{E}[-2]^\bullet \longrightarrow \mathscr{E}^\bullet \longrightarrow ?\ .$$

This triangle is the direct sum of two distinct triangles, $0 \longrightarrow \mathscr{E}^\bullet \longrightarrow \mathscr{E}^\bullet \xrightarrow{+1} 0$ and $\mathscr{E}[-2]^\bullet \longrightarrow \mathscr{E}[-2]^\bullet \longrightarrow 0 \longrightarrow \mathscr{E}[-1]^\bullet$, so

$$\mathscr{E}[-2]^\bullet \longrightarrow \mathscr{E}^\bullet \oplus \mathscr{E}[-2]^\bullet \longrightarrow \mathscr{E}^\bullet \longrightarrow \mathscr{E}[-1]^\bullet$$

therefore $? = \mathscr{E}[-1]^\bullet$, so the element $T_{\mathscr{E}^\bullet}(\mathscr{E}^\bullet)$ has infinite order. Moreover, $\omega$ maps the autoequivalence $T_{\mathscr{E}^\bullet}$ to a suitable riflection. Let us see why. First, we will need to prove something about the behavior of the Mukai vector with respect to spherical objects.

**Lemma 4.1.6.** *If $\mathscr{E}^\bullet \in \mathscr{D}(X)$ is stable for some stability condition on $X$, then*

$$v(\mathscr{E}^\bullet) \geq 2$$

*with equalty precisely when $\mathscr{E}^\bullet$ is spherical.*

*Proof.* The lemma simply follows by Proposition 4.1.2 plus the fact that each object which is stable for some stability condition is simple. But $\mathscr{E}^\bullet$ is simple if and only if $\mathrm{Hom}_{\mathscr{D}(X)}(\mathscr{E}^\bullet, \mathscr{E}^\bullet) = \mathbb{C}$ (Schur's Lemma) and, by Serre duality, one also has $\mathrm{Ext}^2(\mathscr{E}^\bullet, \mathscr{E}^\bullet) = \mathbb{C}$. Therefore, by Proposition **??** :

$$\begin{aligned} v(\mathscr{E}^\bullet)^2 &= (v(\mathscr{E}^\bullet), v(\mathscr{E}^\bullet)) \\ &= \chi(\mathscr{E}^\bullet, \mathscr{E}^\bullet) \\ &= \hom(\mathscr{E}^\bullet, \mathscr{E}^\bullet) - \mathrm{ext}^1(\mathscr{E}^\bullet, \mathscr{E}^\bullet) + \mathrm{ext}^2(\mathscr{E}^\bullet, \mathscr{E}^\bullet) \\ &= 2 - \mathrm{ext}^1(\mathscr{E}^\bullet, \mathscr{E}^\bullet) \end{aligned}$$

therefore $0 \leq \mathrm{ext}^1(\mathscr{E}^\bullet, \mathscr{E}^\bullet) = 2 - v(\mathscr{E}^\bullet)^2$ and $v(\mathscr{E}^\bullet)^2 \geq 2$. If, moreover $\mathscr{E}^\bullet$ is spherical, one has $\mathrm{ext}^1(\mathscr{E}^\bullet, \mathscr{E}^\bullet) = 0$ and therefore $v(\mathscr{E}^\bullet)^2 = 2$.

□

This lemma shows that the Mukai vector of each spherical object lies in the root system:

$$\Delta(X) = \{\delta \in \mathscr{N}(X) \mid (\delta, \delta) = 2\}.$$

**Proposition 4.1.7.** *The twist functor $T_{\mathscr{E}}$ is mapped by $\omega$ to the reflection:*

$$f(v) = v - (v(\mathscr{E}), v)\ v(\mathscr{E})\ .$$

*Proof.*

$$\mathscr{C} := \mathrm{Cone}(\mathscr{E}^\vee \boxtimes \mathscr{E} \longrightarrow \mathscr{O}_\Delta)$$

$$\mathscr{E}^\vee \boxtimes \mathscr{E} \longrightarrow \mathscr{O}_\Delta \longrightarrow \mathscr{C} \xrightarrow{+1} (\mathscr{E}^\vee \boxtimes \mathscr{E})[1]$$

$$\mathrm{ch}(\mathscr{C}) = \mathrm{ch}(\mathscr{O}_\Delta) - \mathrm{ch}(\mathscr{E}^\vee \boxtimes \mathscr{E})$$

By Grothendieck-Riemann-Roch applied to the diagonal embedding $i : X \overset{\cong}{\hookrightarrow} \Delta \subset X \times X$:

$$\mathrm{ch}(\mathscr{O}_\Delta) \cdot \mathrm{Td}(X \times X) = i_*(\mathrm{ch}(\mathscr{O}_X) \cdot \mathrm{Td}(X)) = i_*\mathrm{Td}(X).$$

Dividing by $\sqrt{\mathrm{Td}(X \times X)}$ and using the fact that $i^*\sqrt{\mathrm{Td}(X \times X)} = \mathrm{Td}(X)$ yields:

$$\begin{aligned}
\mathrm{ch}(\mathscr{O}_\Delta) \cdot \sqrt{\mathrm{Td}(X \times X)} &= \mathrm{ch}(\mathscr{O}_\Delta) \cdot \mathrm{Td}(X \times X) \cdot \sqrt{\mathrm{Td}(X \times X)}^{-1} \\
&= i_*\mathrm{Td}(X) \cdot \sqrt{\mathrm{Td}(X \times X)}^{-1} \\
&= i_*(\mathrm{Td}(X) \cdot i^*\sqrt{\mathrm{Td}(X \times X)}^{-1}) \\
&= i_*(\mathrm{Td}(X) \cdot (i^*\sqrt{\mathrm{Td}(X \times X)})^{-1}) \\
&= i_*(\mathrm{Td}(X) \cdot \mathrm{Td}(X)^{-1}) \\
&= i_*(1)\ .
\end{aligned}$$

Hence,

$$\begin{aligned}
&p_*\left(\mathrm{ch}(\mathscr{O}_\Delta) \cdot \sqrt{\mathrm{Td}(X \times X)} \cdot q^*(v)\right) \\
&\quad = p_*(i_*(1) \cdot q^*(v)) = p_*(i_*(i^*q^*(v))) = v\ .
\end{aligned}$$

Finally, using the fact that $\mathrm{ch}(\mathscr{O}_\Delta)$ is an even cohomology class, so it commutes with the first term, (?) one can compute the algebraic cycle corresponding to $\mathscr{E}$:

$$\begin{aligned}
Z &= q^*\sqrt{Td(X)}\mathrm{ch}(\mathscr{C})p^*\sqrt{Td(X)} \\
&= q^*\sqrt{Td(X)}\mathrm{ch}(\mathscr{O}_\Delta)p^*\sqrt{Td(X)} - q^*\sqrt{Td(X)} \cdot \mathrm{ch}(\mathscr{E}^\vee \boxtimes \mathscr{E}) \cdot p^*\sqrt{Td(X)} \\
&= \mathrm{ch}(\mathscr{O}_\Delta)q^*\sqrt{Td(X)}p^*\sqrt{Td(X)} - q^*\sqrt{Td(X)} \cdot \mathrm{ch}(q^*\mathscr{E}^\vee \otimes p^*\mathscr{E}) \cdot p^*\sqrt{Td(X)} \\
&= \mathrm{ch}(\mathscr{O}_\Delta) \cdot \sqrt{\mathrm{Td}(X \times X)} - q^*\sqrt{Td(X)} \cdot \mathrm{ch}(q^*\mathscr{E}^\vee) \cdot \mathrm{ch}(p^*\mathscr{E}) \cdot p^*\sqrt{Td(X)} \\
&= \mathrm{ch}(\mathscr{O}_\Delta) \cdot \sqrt{\mathrm{Td}(X \times X)} - q^*\sqrt{Td(X)} \cdot q^*\mathrm{ch}(\mathscr{E}^\vee) \cdot p^*\mathrm{ch}(\mathscr{E}) \cdot p^*\sqrt{Td(X)} \\
&= \mathrm{ch}(\mathscr{O}_\Delta) \cdot \sqrt{\mathrm{Td}(X \times X)} - q^*(\mathrm{ch}(\mathscr{E}^\vee) \cdot \sqrt{Td(X)}) \cdot p^*(\mathrm{ch}(\mathscr{E}) \cdot \sqrt{Td(X)}) \\
&= \mathrm{ch}(\mathscr{O}_\Delta) \cdot \sqrt{\mathrm{Td}(X \times X)} - q^*v(\mathscr{E}^\vee) \cdot p^*v(\mathscr{E})
\end{aligned}$$

$$\begin{aligned}
f(v) &= p_*(Z \cdot q^*(v)) \\
&= p_*(\mathrm{ch}(\mathscr{O}_\Delta) \cdot \sqrt{\mathrm{Td}(X \times X)} \cdot q^*(v)) - p_*(q^* v(\mathscr{E}^\vee) \cdot p^* v(\mathscr{E}) \cdot q^*(v)) \\
&= v - p_* q^*(v(\mathscr{E}^\vee) \cdot v) v(\mathscr{E}) \\
&= v - (\int_X v(\mathscr{E})^\star \cdot v) \cdot v(\mathscr{E}) \\
&= v - (v(\mathscr{E}), v)\ v(\mathscr{E})\ .
\end{aligned}$$

Thus the functor $T^2_{\mathscr{E}^\bullet}$ defines an element of $\mathrm{Aut}^0 \mathscr{D}(X) := \mathrm{Ker}\omega$.

□

We now want to examine the action of the group $\mathrm{Aut}\mathscr{D}(X)$ on the space of stability conditions $\mathrm{Stab}(X)$. A stability condition is said to be *numerical* if its central charge takes the form

$$Z(\mathscr{E}^\bullet) = (\pi(\sigma), v(\mathscr{E}^\bullet))$$

for some vector $\pi(\sigma) \in \mathscr{N}(X) \otimes \mathbb{C}$, where

$$\pi : \mathrm{Stab}(X) \longrightarrow \mathscr{N}(X) \otimes \mathbb{C}$$

is the map sending a stability condition to its central charge, which is continuous. To describe the image of this map, define an open subset:

$$\mathscr{P}(X) \subset \mathscr{N}(X) \otimes \mathbb{C}$$

consisting of those vectors whose real and imaginary parts span a positive-definite two-planes in $\mathscr{N}(X) \otimes \mathbb{R}$, i.e.

$$\mathscr{P}(X) := \{v \in \mathscr{N}(X) \otimes \mathbb{C} \mid (\bullet, \bullet)|_{\mathrm{Span}_{\mathbb{R}}\{\Re v, \Im v\}} \text{ is positive definite } \}$$

This space has two connected components, which are exchanged by complex conjugation. Notice that $GL^+(2, \mathbb{R})$ acts freely on $\mathscr{P}(X)$ by simply identifying $\mathscr{N}(X) \otimes \mathbb{C}$ with $\mathscr{N}(X) \otimes \mathbb{R}$. A section of this action is provided by the submanifold:

$$\mathscr{Q}(X) = \{v \in \mathscr{P}(X) \mid (v, v) = 0,\ (v, \bar{v}) > 0 \text{ and } r(v) = 1\}$$

which can be identified with

$$\{D + iF \in \mathrm{NS}(X) \otimes \mathbb{C} \mid F^2 > 0\}$$

via the exponential map

$$D + iF \mapsto v = e^{D+iF} = \left(1, D + iF, \frac{1}{2}(D^2 - F^2) + i(D \cdot F)\right).$$

Now, let $\mathscr{P}^+(X) \subset \mathscr{P}(X)$ be the connected component containing the image of the exponential map for $F \in \mathrm{NS}(X) \otimes \mathbb{R}$ ample. For each $\delta \in \Delta(X)$ let

$$\delta^\perp = \{v \in \mathscr{N}(X) \otimes \mathbb{C} \mid (v, \delta) = 0\} \subset \mathscr{N}(X) \otimes \mathbb{C}.$$

We state one of the fundamental theorems of the theory:

**Theorem 4.1.8.** *There is a connected component* $\mathrm{Stab}^\bullet(X) \subset \mathrm{Stab}(X)$ *which is mapped by* $\pi$ *onto the open subset*

$$\mathscr{P}_0^+(X) = \mathscr{P}^+(X) \setminus \bigcup_{\delta \in \Delta(X)} \delta^\perp \subset \mathscr{N}(X) \otimes \mathbb{C}.$$

*Moreover, the induced map*

$$\pi|_{\mathrm{Stab}^\bullet(X)} : \mathrm{Stab}^\bullet(X) \longrightarrow \mathscr{P}_0^+(X)$$

*is a covering map, and the subgroup of* $\mathrm{Aut}^0\mathscr{D}(X)$ *which preserves the connected component* $\mathrm{Stab}^\bullet(X)$ *acts freely on it and is the group of deck transformations of* $\pi$*.*

Finally, one would like to describe the group $\mathrm{Aut}\mathscr{D}(X)$. An argument by Orlov shows that the image of the map $\omega$ contains the index-two subgroup

$$\mathrm{Aut}^+ H^*(X, \mathbb{Z}) \subset \mathrm{Aut} H^*(X, \mathbb{Z})$$

consisting of the elements that do not exchange the connected components of $\mathscr{P}(X)$. The following conjecture gives an interesting characterization of all the objects involved:

**Conjecture 4.1.9.** *The action of the group* $\mathrm{Aut}\mathscr{D}(X)$ *on* $\mathrm{Stab}(X)$ *preserves the connected component* $\mathrm{Stab}^\bullet(X)$*. This connected component is simply connected, and therefore there is a short exact sequence of groups:*

$$1 \longrightarrow \pi_1(\mathscr{P}_0^+(X)) \longrightarrow \mathrm{Aut}\mathscr{D}(X) \xrightarrow{\omega} \mathrm{Aut}^+ H^*(X, \mathbb{Z}) \longrightarrow 1.$$

We will now give some results which will allow us to describe the wall-and-chamber structure of $\mathrm{Stab}(X)$.

**Lemma 4.1.10.** *Let* $\mathscr{A} \subset \mathscr{D}(X)$ *be the heart of a boundet t-structure on* $\mathscr{D}(X)$*. Then if*

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

*is a short exact sequence in* $\mathscr{A}$ *such that* $\mathrm{Hom}_{\mathscr{D}(X)}(A,C)=0$*, the following inequality holds:*

$$\mathrm{ext}^1(A,A)+\mathrm{ext}^1(C,C)\leq \mathrm{ext}^1(B,B)\ .$$

*Proof.* Given $E,F\in\mathscr{A}$, write

$$(E,F)^i := \mathrm{ext}^i(E,F).$$

Then $(E,F)^i=0$ unless $i=0,1,2$ and, by Serre duality , $(E,F)^i=(F,E)^{2-i}$. Recall that the Euler bilinear form splits on short exact sequences, i.e.

$$\chi(A,X)+\chi(C,X)=\chi(B,X)$$

for each $X\in\mathscr{A}$. We make the computation when $X=A,B,C$:

$$\begin{gathered}
2(A,A)^0-(A,A)^1+(C,A)^0-(C,A)^1+(A,C)^0=(B,A)^0-(B,A)^1+(A,B)^0\\
(A,C)^0-(A,C)^1+(C,A)^0+2(C,C)^0-(C,C)^1=(B,C)^0-(B,C)^1+(C,B)^0\\
(A,B)^0-(A,B)^1+(B,A)^0+(C,B)^0-(C,B)^1+(B,C)^0=2(B,B)^0-(B,B)^1
\end{gathered}$$

then, summing up and considering that $(A,C)^0=0$, we get:

$$\begin{aligned}
&2(A,A)^0-(A,A)^1+(C,A)^0-(C,A)^1-(A,C)^1+(C,A)^0+2(C,C)^0-\\
&\quad-(C,C)^1+(A,B)^0-(A,B)^1+(B,A)^0+(C,B)^0-(C,B)^1+(B,C)^0=\\
&=(B,A)^0-(B,A)^1+(A,B)^0+(B,C)^0-(B,C)^1+(C,B)^0+2(B,B)^0-(B,B)^1
\end{aligned}$$

which gives

$$\begin{aligned}
&2(A,A)^0+2(C,A)^0-2(C,A)^1+2(C,C)^0-(A,A)^1-(C,C)^1=\\
&\qquad=2(B,B)^0-(B,B)^1\leq 2(B,B)^0-(A,A)^1-(C,C)^1
\end{aligned}$$

and finally

$$(A,A)^0+(C,A)^0-(C,A)^1+(C,C)^0\leq (B,B)^0\ .$$

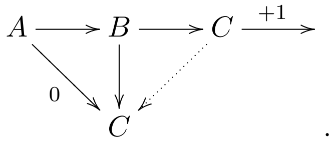
This is equivalent to the inequality we want to prove, and follows by the existence of the short exact sequence

$$0\longrightarrow \mathrm{Hom}(C,A)\longrightarrow \mathrm{End}(B)\longrightarrow \mathrm{End}(A)\oplus\mathrm{End}(C)\longrightarrow \mathrm{Ext}^1(C,A)$$

to show where it comes from, we can observe that, by the hypotesis $(A,C)^0$, each endomorphism of $B$ induces an endomorphism of the entire triangle $A\longrightarrow B\longrightarrow C\stackrel{+1}{\longrightarrow}$, seen as a triangle in the whole derived category. Actually, one has:

$$\begin{array}{ccccccc}
A & \xrightarrow{\alpha} & B & \xrightarrow{\beta} & C & \xrightarrow{+1} & \\
 & & \downarrow{\scriptstyle f} & \searrow & & & \\
A & \xrightarrow{\alpha} & B & \xrightarrow{\beta} & C & \xrightarrow{+1} &
\end{array}$$

where the dotted arrow is just the composition. But this means that an arrow in $\mathrm{End}(C)$ is induced:



Moreover, by the third axiom of triangulated categories, a couple of maps $B \longrightarrow B$ and $C \longrightarrow C$ can be completed to a map of the whole triangle, i.e., an arrow $A \longrightarrow A$ is induced. $\square$

Recall Theorem 2.4.8:

**Theorem 4.1.11.** *For each connected component* $\mathrm{Stab}^*(X) \subset \mathrm{Stab}(X)$ *there exists a linear subspace* $V \subset \mathscr{N}(X) \otimes \mathbb{C}$ *such that the map*

$$\pi : \mathrm{Stab}^*(X) \longrightarrow \mathscr{N}(X) \otimes \mathbb{C}$$

*is a local homeomorphism on an opens subset of the subspace* $V$*.*

In all known examples, the subspace $V$ is actually the whole $\mathscr{N}(X) \otimes \mathbb{C}$, but the fact that a general result has not been proved jet, is useful to give the following definition:

**Definition 4.1.12.** A connected component $\mathrm{Stab}^*(X) \in \mathrm{Stab}(X)$ of the space of stability condition is called *full* if the subspace in the theorem above is the whole $\mathscr{N}(X) \otimes \mathbb{C}$. A stability condition $\sigma \in \mathrm{Stab}^*(X) \subset \mathrm{Stab}(X)$ is called *full* if $\sigma \in \mathrm{Stab}^*(X)$ for some full connected component $\mathrm{Stab}^*(X)$.

Notice that full stability conditions are preserved by autoequivalences.
We now give a definition whose utility will appear later.

**Definition 4.1.13.** A stability condition $\sigma = (Z, \mathscr{P}) \in \mathrm{Stab}(X)$ is called *discrete* if the image of the central charge $Z : K(X) \longrightarrow \mathbb{C}$ is a discrete subgroup.

**Lemma 4.1.14.** *Fix* $0 < \varepsilon < \frac{1}{2}$*, and let* $\sigma = (Z, \mathscr{P})$ *be discrete. Then for all* $\phi \in \mathbb{R}$*, the quasi-abelian category* $\mathscr{P}((\phi - \varepsilon, \phi + \varepsilon))$ *is of finite length. In particular,* $\sigma$ *is locally finite.*

*Proof.* Fix $\phi \in \mathbb{R}$ and call $\mathscr{A} := \mathscr{P}((\phi - \varepsilon, \phi + \varepsilon))$. The central charge of an object $A \in \mathscr{A}$, by definition, lies in the sector

$$S = \{z = re^{i\pi\phi} \mid r > 0 \text{ and } \phi - \varepsilon < \psi < \phi + \varepsilon\}$$

because it sends in $S$ all the semistable factors in the Harder-Narasimhan filtration of $A$. But $S$ is strictly contained in a half-plane of $\mathbb{C}$, because we have chosen $\varepsilon$ to vary in $(0, \frac{1}{2})$. Define a real-valued function

$$\begin{aligned} f : K(X) &\longrightarrow \mathbb{R} \\ A &\mapsto \Re(e^{-i\pi\phi} Z(A)) \quad . \end{aligned}$$

Then $f(A) > 0$ for all nonzero $A$. Indeed

$$\begin{aligned} f(A) &= \Re(e^{-i\pi\phi}Z(A)) \\ &= \Re(e^{-i\pi\phi}r(A)e^{i\pi\phi(A)}) \\ &= \Re(r(A)e^{i\pi(\phi(A)-\phi}) \end{aligned}$$

but $\phi(A) - \phi \in (-\varepsilon, \varepsilon) \subset (-\frac{1}{2}, \frac{1}{2})$, therefore $e^{i\pi(\phi(A)-\phi}$ lies in the first quadrant of the complex plane, which implies that $f(A) > 0$.
Moreover, $f$ splits on strict short exact sequences in $\mathscr{A}$: if $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ is a strict short exact sequence in $\mathscr{A}$, then $f(B) = f(A) + f(C)$. Indeed we have that:

$$\begin{aligned} &Z(B) = Z(A) + Z(C); \\ &e^{-i\pi\phi}Z(B) = e^{-i\pi\phi}Z(A) + e^{-i\pi\phi}Z(C); \\ &\Re(e^{-i\pi\phi}Z(B)) = \Re(e^{-i\pi\phi}Z(A)) + \Re(e^{-i\pi\phi}Z(C)); \\ &f(B) = f(A) + f(C). \end{aligned}$$

Therefore, for each $E \in \mathscr{A}$, the central charges of its subobjects and quotients lie in the bounded region

$$\{z \in S \mid \Re(e^{-i\pi\phi}z) < f(E)\} \ .$$

The hypoteses that $\sigma$ is discrete implies that there is only a finite number of possibilities and, by the fact that each subobject must be sent by $f$ to a strictly smaller point belonging to the segment $(0, f(E))$, each chain of subobjects must be finite (same goes for quotients). Therefore, $\sigma$ is locally finite.

□

**Lemma 4.1.15.** *Let $\sigma \in \mathrm{Stab}(X)$ be a full stability condition, and fix $0 < \varepsilon < \frac{1}{2}$ as above. Then for each $\phi \in \mathbb{R}$ the quasi-abelian category $\mathscr{P}((\phi - \varepsilon, \phi + \varepsilon))$ is of finite length.*

## 4.2 Constructing stability conditions

Recall, by the previous chapter, that the classical slope function for torsion-free sheaves $\mathscr{F}$ is given by:

$$\mu_H(\mathscr{F}) = \frac{c_1(\mathscr{F}) \cdot H}{\mathrm{rk}(\mathscr{F})}$$

for each ample divisor $H$. Therefore, the notion of stability strongly depends on the choice of an ample divisor on the surface. The associated notion of semistability is the usual one, i.e. a torsion-free sheaf $\mathscr{F}$ is semistable if for each proper subsheaf $\mathscr{G} \subset \mathscr{F}$, one has $\mu_H(\mathscr{G}) \leq \mu_H(\mathscr{F})$. Therefore, we can build the Harder-Narasimhan filtration with respect to $H$ for each coherent sheaf $\mathscr{E}$:

$$\mathscr{E}_0 \subset \mathscr{E}_1 \subset ... \subset \mathscr{E}_n = \mathscr{E}$$

where:

1. $\mathscr{E}_0 = \text{tors}(\mathscr{E})$;
2. For each $i = 1, ..., n$, the sheaf $\mathscr{F}_i := \mathscr{E}_i/\mathscr{E}_{i-1}$ is torsion free and $H$-semistable;
3. The slopes form a strictly decreasing sequence: $\mu_H^+(\mathscr{E}) := \mu_H(\mathscr{F}_1) > \mu_H(\mathscr{F}_2) > ... > \mu_H(\mathscr{F}_n) =: \mu_H^-(\mathscr{E})$.

The existence and the uniqueness of the Harder-Narasimhan filtration have been proved in the previous chapter.

*Remark* 4.2.1. For any torsion sheaf $\mathscr{G}$, the H-N filtration is $0 \subset \mathscr{G}$. Therefore, there are no semistable quotients we can use to compute slopes, and, by convention, we set $\mu_H(\mathscr{G}) = +\infty$.

*Remark* 4.2.2. We could of course associate to the slope function defined above a family of group homomorphisms on the abelian category $\text{Coh}(X)$:

$$Z_H : K(\text{Coh}(X)) \longrightarrow \mathbb{C}$$
$$\mathscr{E} \mapsto r(\mathscr{E})e^{i\pi\mu_H(\mathscr{E})}$$

for each torsion-free sheaf $\mathscr{E}$ (of course an arbitrary coherent sheaf will be mapped to the sum of the images of its semistable torsion-free quotients), each one given by a different choice of $r(\mathscr{E})$. However, notice that this one cannot be a Bridgeland stability function: in fact, by definition, the positive cone should be mapped to the upper-half plane $\mathbb{H}$, thus the image of each nonzero object should be a vector of $\mathbb{H}$, i.e., with nonzero slope. But the group homomorphism defined above sends the sheaf $\mathscr{O}_S$ to a point on the positive half of the real line ($c_1(\mathscr{O}_S) = 0$, therefore $\mu_H(\mathscr{O}_S) = 0$, too) which does not belong to $\mathbb{H}$.

Let us now outline a strategy to build a class of Bridgeland stability conditions and to study semistable objects. First, we need to take a pair of $\mathbb{R}$- divisors $D, F$, with $F$ ample. Consider the following group homomorphism:

$$Z_{(D,F)} : K(\text{Coh}(X)) \longrightarrow \mathbb{C}$$

such that

$$Z_{(D,F)}([\mathscr{E}]) := -\int_S e^{-(D+iF)} ch([\mathscr{E}])\sqrt{Td(X)}$$

for each $E \in \text{Coh}(X)$. We can extend it to the Grothendieck's group of the whole derived category, $K(\mathscr{D}(X))$, by simply setting:

$$Z_{(D,F)}([\mathscr{E}^\bullet]) := \sum_i (-1)^i Z_{(D,F)}(H^i(\mathscr{E}^\bullet)).$$

Let us write the central charge in a more explicit way[6]. As we know, the expression $e^{-(D+iF)}$ stands for the vector $(1, -(D+iF), \frac{1}{2}(D^2-F^2)+iD\cdot F) = (1, D+iF, \frac{1}{2}(D^2-F^2)+iD\cdot F)^*$. Therefore the central charge $Z_{(D,F)}$ simply reads:

$$Z_{(D,F)}(\mathscr{E}^\bullet) = -(e^{D+iF}, v(\mathscr{E}^\bullet)).$$

Therefore, for an arbitrary complex $\mathscr{E}^\bullet$, supposing that $v(\mathscr{E}^\bullet) = (r, \Delta, s)$ one has:

$$\begin{aligned} Z(\mathscr{E}^\bullet) &= ((1, -D+iF, \frac{1}{2}(D^2-F^2)+iD\cdot F), (r, \Delta, s)) \\ &= ((D+iF)\cdot\Delta - s - \frac{r}{2}(D^2-F^2) - riD\cdot F \\ &= D\cdot\Delta + iF\cdot\Delta - s - \frac{r}{2}(D^2-F^2) - irD\cdot F \\ &= D\cdot\Delta - \frac{r}{2}(D^2-F^2) - s + iF(\Delta - rD). \end{aligned}$$

If $r \neq 0$:

$$\begin{aligned} Z(\mathscr{E}^\bullet) &= \frac{1}{2r}(2rD\cdot\Delta - r^2(D^2-F^2) - 2rs) + iF(\Delta - rD) \\ &= \frac{1}{2r}(2rD\cdot\Delta - r^2D^2 - \Delta^2 + \Delta^2 + r^2F^2 - 2rs) + iF(\Delta - rD) \\ &= \frac{1}{2r}((\Delta - 2rs)^2 + r^2F^2 - (\Delta - rD)^2) + iF(\Delta - rD) \end{aligned}$$

while if $r = 0$, just the term $(D\cdot\Delta - s) + iF\cdot(\Delta - rD)$ survives.
Second, we need a slicing which must be compatible with $Z_{(D,F)}$ in order to create a Bridgeland slope function. We already know, by Proposition 2.3.5, that a stability condition can be given as a pair $(Z, \mathscr{P})$, where $Z$ is a central charge and $\mathscr{P}$ is a slicing compatible with $Z$, as well as a pair $(Z, \mathscr{A})$, where $\mathscr{A}$ is the heart of a bounded t-structure on the derived category $\mathscr{D}(S)$: we choose, in this case, to look for the heart of a suitable t-structure, and we will build it as a tilt. If $\mathscr{A}$ is the heart of a t-strutcure on a given triangulated category $\mathscr{D}$, we can define its *right* and *left tilt* respectively as:

$$\mathscr{A}^\# := \{E \in \mathscr{D} \mid H^i_{\mathscr{A}}(E) = 0 \ \forall i \neq -1, 0 \ , \ H^0_{\mathscr{A}}(E) \in \mathscr{T}, \ H^{-1}_{\mathscr{A}}(E) \in \mathscr{F}\}$$

$$\mathscr{A}^b := \{E \in \mathscr{D} \mid H^i_{\mathscr{A}}(E) = 0 \ \forall i \neq 0, 1 \ , \ H^0_{\mathscr{A}}(E) \in \mathscr{T}, \ H^1_{\mathscr{A}}(E) \in \mathscr{F}\}$$

where $(\mathscr{T}, \mathscr{F})$ is a torsion pair, i.e. a pair of full additive subcategories $\mathscr{T}, \mathscr{F} \subset \mathscr{A}$ such that:

1. $\mathrm{Hom}(T, F) = 0$ for each $T \in \mathscr{T}, \ F \in \mathscr{F}$;

[6] Notice that here we have dropped square parentheses because we did not want notation to be too heavy; however, from now on we will write $\mathscr{E}^\bullet$ to indicate its class $[\mathscr{E}^\bullet] \in K(X)$, unless differently specified.

2. For each $E \in \mathscr{A}$, there exists a distinguished triangle $T \longrightarrow E \longrightarrow F \xrightarrow{+1}$ such that $T \in \mathscr{T}$, $F \in \mathscr{F}$.

In our case, we want to tilt with respect to the heart of the trivial t-structure $\mathscr{A} := \mathrm{Coh}(X)$. We define a suitable torsion pair as:

$$\mathscr{T} := \{\text{torsion sheaves}\} \cup \{\mathscr{E} \in \mathscr{A} \mid \mu_H(\mathscr{E}) > D \cdot F\}$$
$$\mathscr{F} := \{\mathscr{E} \in \mathscr{A} \mid \mu_H(\mathscr{E}) \leq D \cdot F\}$$

and the corresponding tilted heart will be:

$$\mathscr{A}^{\#}_{(D,F)} := \{\mathscr{E}^\bullet \in \mathscr{D}(S) \mid H^i(\mathscr{E}^\bullet) = 0 \ \forall i \neq -1, 0 \ , \ H^{-1}(\mathscr{E}^\bullet) \in \mathscr{F}, \ H^0(\mathscr{E}^\bullet) \in \mathscr{T}\}$$

We want to show first that the pair $(Z_{(D,F)}, \mathscr{A}^{\#}_{(D,F)})$ makes a Bridgeland slope function. To prove that it actually is a stability condition, we will need to show that the central charge has the HN property, which will require some additional work.

**Proposition 4.2.3.** *The pair* $(Z_{(D,F)}, \mathscr{A}^{\#}_{(D,F)})$ *is a Bridgeland slope function.*

*Proof.* We need to show that for each $\mathscr{E}^\bullet \in \mathscr{A}^{\#}_{(D,F)}$ one has that $Z_{(D,F)}(\mathscr{E}^\bullet) \in \mathbb{H}$. Notice that if $\mathscr{E}^\bullet \in \mathscr{A}^{\#}_{(D,F)}$ there is a short exact sequence (in $\mathscr{A}^{\#}_{(D,F)}$):

$$0 \longrightarrow H^{-1}(\mathscr{E}^\bullet)[1] \longrightarrow \mathscr{E}^\bullet \longrightarrow H^0(\mathscr{E}^\bullet) \longrightarrow 0 \ .$$

In fact, suppose that the equivalence class $E$ in the derived category $\mathscr{D}(S)$ is represented by the complex:

$$\mathscr{E}^\bullet = \{... \xrightarrow{d^{-3}} \mathscr{E}^{-2} \xrightarrow{d^{-2}} \mathscr{E}^{-1} \xrightarrow{d^{-1}} \mathscr{E}^0 \xrightarrow{d^0} \mathscr{E}^1 \xrightarrow{d^1} ...\}$$

Then the fact that $\mathscr{E}$ belongs to $\mathscr{A}^{\#}$ implies that $E$ is isomorphic to a complex which looks like the following one:

$$... \longrightarrow 0 \longrightarrow ...\mathrm{Coker} d^{-2} \xrightarrow{d^{-2}} \mathscr{E}^{-1} \xrightarrow{d^{-1}} \mathscr{E}^0 \xrightarrow{d^0} \mathrm{Im} d^0 \xrightarrow{d^1} 0 \longrightarrow ...$$

Then the complex $H^{-1}(\mathscr{E}^\bullet)[1]$ is isomorphic to the complex

$$... \longrightarrow 0 \longrightarrow ...\mathrm{Coker} d^{-2} \xrightarrow{d^{-2}} \mathscr{E}^{-1} \xrightarrow{d^{-1}} \mathrm{Im} d^{-1} \longrightarrow 0 \longrightarrow ...$$

which naturally embeds in $\mathscr{E}^\bullet$:

$$\begin{array}{ccccccccccc}
... & \longrightarrow & \mathrm{Coker} d^{-2} & \xrightarrow{d^{-2}} & \mathscr{E}^{-1} & \xrightarrow{d^{-1}} & \mathrm{Im} d^{-1} & \xrightarrow{d^0} & 0 & \longrightarrow & 0 & \longrightarrow & ... \\
 & & \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle i} & & \downarrow & & \downarrow & & \\
... & \longrightarrow & \mathrm{Coker} d^{-2} & \xrightarrow{d^{-2}} & \mathscr{E}^{-1} & \xrightarrow{d^{-1}} & \mathscr{E}^0 & \xrightarrow{d^0} & \mathrm{Im} d^0 & \xrightarrow{d^1} & 0 & \longrightarrow & ...
\end{array}$$

- The first square obviously commutes;
- The second square commutes by definition: indeed, each of the two ways lead to $\mathrm{Im} d^{-1}$ by $d^{-1}$;

- The third square commutes by exactness: one of the first composition is obviously zero, and the other one is zero, too, because the image of $d^{-1}$ is contained in the kernel of $d^0$.

The quotient is simply given by the complex

$$\begin{array}{ccccccccccccc}
\dots & \longrightarrow & \mathrm{Coker}d^{-2} & \xrightarrow{d^{-2}} & \mathscr{E}^{-1} & \xrightarrow{d^{-1}} & \mathrm{Im}d^{-1} & \xrightarrow{d^{0}} & 0 & \longrightarrow & 0 & \longrightarrow & \dots \\
 & & \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle i} & & \downarrow & & \downarrow & & \\
\dots & \longrightarrow & \mathrm{Coker}d^{-2} & \xrightarrow{d^{-2}} & \mathscr{E}^{-1} & \xrightarrow{d^{-1}} & \mathscr{E}^{0} & \xrightarrow{d^{0}} & \mathrm{Im}d^{0} & \xrightarrow{d^{1}} & 0 & \longrightarrow & \dots \\
 & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \\
\dots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \mathrm{Coker}d^{-1} & \xrightarrow{f} & \mathrm{Im}d^{0} & \xrightarrow{g} & 0 & \longrightarrow & \dots
\end{array}$$

whose cohomology is exact the one we wanted: indeed, if we call $\mathscr{Q}^\bullet$ the quotient complex, then

$$H^0(\mathscr{Q}^\bullet) = \mathrm{Ker}f = \mathrm{Ker}d^0/\mathrm{Im}d^{-1} \cong H^0(\mathscr{E}^\bullet)$$

$$H^1(\mathscr{Q}^\bullet) = \mathrm{Im}d^0/\mathrm{Im}f = \mathrm{Im}d^0/\mathrm{Im}d^0 = 0$$

therefore, the complex $Q$ is quasi-isomorphic to the complex $\{\dots \longrightarrow H^0(\mathscr{E}^\bullet) \longrightarrow \dots\} = H^0(\mathscr{E}^\bullet)$.
Now, by the fact that distinct triangles split in $K(X)$ and that $Z$ is a group homomorphism, it will suffices to show that:

1. If $\mathscr{E} \in \mathscr{F}$, then $Z(\mathscr{E}[1]) \in \mathbb{H}$;
2. if $\mathscr{E} \in \mathscr{T}$, then $Z(\mathscr{E}) \in \mathbb{H}$.

*Remark* 4.2.4. Recall that $Z(\mathscr{E}[1]) = -Z(\mathscr{E})$!

If $\mathscr{E} \in \mathscr{T}$, then either $\mathscr{E}$ is torsion or the slope of its semistable quotients is strictly greater then $D \cdot F$. For what concerns the point 2), therefore, we need to distinguish some cases:

$\mathscr{E}$ **is torsion,** $\dim \mathbf{Supp}(\mathscr{E}) = 1$ . Then

$$r(\mathscr{E}) = 0 \ \Rightarrow \ Z(\mathscr{E}) = (C_1(\mathscr{E}) \cdot D - s) + i(c_1(\mathscr{E}) \cdot F).$$

But $c_1(\mathscr{E}) \cdot F > 0$ because $c_1(\mathscr{E})$ is effective, therefore $Z(\mathscr{E})$ lies in the upper-half plane.

$\mathscr{E}$ **is torsion,** $\dim \mathbf{Supp}(\mathscr{E}) = 0$ . Then $\Im Z(\mathscr{E}) = 0$, because $c_1(\mathscr{E}) = 0$. So we just need to check that the real part of $Z(\mathscr{E})$ is negative. But:

$$\Re Z(\mathscr{E}) = -s = -\mathrm{ch}_2(\mathscr{E}) = c_2(\mathscr{E})$$

$\mathscr{E}$ **is not torsion** . Then we can assume $\mathscr{E}$ to be torsion free, as it suffices to show that each of the semistable quotients of $\mathscr{E}$ lie in the upper-half plane. We are now able to consider the slope of $\mathscr{E}$ with respect to F, which is subjected to the condition:

$$\mu_F(\mathscr{E}) > D \cdot F \ .$$

But we know that $\mu_F(\mathscr{E}) = \frac{c_1(\mathscr{E})\cdot F}{r(\mathscr{E})}$, therefore the condition becomes $c_1(\mathscr{E}) \cdot F - r(\mathscr{E})D \cdot F > 0$, i.e. $(c_1(\mathscr{E}) - r(\mathscr{E})D) \cdot F > 0$. By the explicit formula which gives the central charge, $(c_1(\mathscr{E}) - r(\mathscr{E})D) \cdot F = \Im Z(\mathscr{E})$, which implies that $Z(\mathscr{E})$ lies in the upper-half plane.

If we now suppose $\mathscr{E} \in \mathscr{F}$ to be torsion-free again (the motivation is exactly the same), we have two cases:

$\mu_F(\mathscr{E}) < D \cdot F$ . Then, as above, $\frac{c_1(\mathscr{E})\cdot D}{r(\mathscr{E})} < D \cdot F$, which equals

$$c_1(\mathscr{E}) \cdot F - r(\mathscr{E})D \cdot F = (c_1(\mathscr{E}) - r(\mathscr{E})D) \cdot F < 0 \ .$$

Therefore $\Im Z(\mathscr{E}) < 0$ and $\Im Z(\mathscr{E}[1]) = -\Im Z(\mathscr{E}) > 0$

$\mu_F(\mathscr{E}) = D \cdot F$ . Again, we have $\Im Z(\mathscr{E}[1]) = 0$, and by Hodge Index Theorem, using the fact that $F$ is ample:

$$(c_1(\mathscr{E}) - r(\mathscr{E})D) \cdot F = 0 \implies (c_1(\mathscr{E}) - r(\mathscr{E})D)^2 \leq 0.$$

□

## 4.3 The covering map

The following lemma is slightly technical, so we will not prove it, but it is useful.

**Lemma 4.3.1.** *Let $||\cdot||$ be a norm on the complexified numerical Grothendieck group $\mathscr{N}(X) \otimes \mathbb{C}$. Then, for each vector $\alpha \in \mathscr{P}(X)$ there exists a constant $r_\alpha > 0$ such that*

$$|(u, v)| \leq r_\alpha ||u|| |(\alpha, v)|$$

*for each $u \in \mathscr{N}(X) \otimes \mathbb{C}$ and $v \in \mathscr{N}(X) \otimes \mathbb{R}$ such that $(v, v) \geq 0$. If $\alpha \in \mathscr{P}_0(X)$, moreover, one can choose $r_\alpha$ such that the inequality holds for each $u \in \mathscr{N}(X) \otimes \mathbb{C}$ and for each $v \in \Delta(X)$.*

**Lemma 4.3.2.** *If $\alpha \in \mathscr{P}_0(X)$ and $m > 0$ is a constant, then there are only finitely many elements $v \in \mathscr{N}(X) such that v^2 \geq 2$ and $|(\alpha, v)| \leq m$.*

**Proposition 4.3.3.** *The subset $\mathscr{P}_0(X) \subset \mathscr{N}(X) \otimes \mathbb{C}$ is open, and the restriction of the natural map:*

$$\pi|_{\pi^{-1}(\mathscr{P}_0(X))} : \pi^{-1}(\mathscr{P}_0(X)) \longrightarrow \mathscr{P}_0(X)$$

*is a covering map.*

*Proof.* There are two assertions to prove:

$\mathscr{P}_0(X)$ **is open** . What we want to see is that for each vector $\alpha \in \mathscr{P}_0(X)$ there is an open subset which contains $\alpha$ and which is itself contained in $\mathscr{P}_0(X)$. Let us fix a norm $||\cdot||$ on the finite-dimensional vector space $\mathscr{N}(X)\otimes\mathbb{C}$, and consider a vector $\alpha \in \mathscr{P}_0(X)$. Lemma 4.3.1 assures us that there exists $r_\alpha > 0$ such that the inequality

$$|(u,v)| \leq r_\alpha ||u|| |(\alpha, v)|$$

holds for each $u \in \mathscr{N}(X)\otimes\mathbb{C}$ and for each $v \in \Delta(X)$ such that $(v,v) \geq 0$. Now for each $\varepsilon > 0$ define the subset:

$$B_\varepsilon(\alpha) = \left\{\beta \in \mathscr{N}(X)\otimes\mathbb{C} \mid ||\beta-\alpha|| < \frac{\varepsilon}{r_\alpha}\right\} \subset \mathscr{N}(X)\otimes\mathbb{C}$$

which is obviously open in $\mathscr{N}(X)\otimes\mathbb{C}$. We want to show that, for a suitable choice of $\varepsilon > 0$, the subset $B_\varepsilon(\alpha) \subset \mathscr{P}_0(X)$. Take $\beta \in B_\varepsilon(\alpha)$. Then one has:

$$\begin{aligned} |(\beta,v)-(\alpha,v)| &= |(\beta-\alpha,v)| \\ &\leq r_\alpha ||\beta-\alpha|| |(\alpha,v)| \\ &< \varepsilon |(\alpha,v)| \end{aligned}$$

for each $v \in \Delta(X)$. This means that if $\varepsilon < 1$, then $\beta$ spans a positive-definite two-plane in $\mathscr{N}(X)\otimes\mathbb{R}$ and $(\beta,v) \neq 0$ for each $v \in \Delta(X)$. Therefore $B_\varepsilon(\alpha) \subset \mathscr{P}_0(X)$.

$\pi|_{\pi^{-1}(\mathscr{P}_0(X))}$ **is a covering map** . Take $\sigma \in \mathit{Stab}(X)$ such that $\pi(\sigma) = \alpha$. We can define a subset

$$C_\varepsilon(\sigma) = \left\{\tau \in \pi^{-1}\left(B_\varepsilon(\alpha)\right) \mid d(\sigma,\tau) < \frac{1}{2}\right\} \subset \mathrm{Stab}(X)$$

where $d$ is the distance we defined in the second chapter on the space $\mathrm{Stab}(X)$. Now, Lemma 4.1.6 tells us that if $\mathscr{E}^\bullet \in \mathscr{D}(X)$ is a stable object, then $(v(\mathscr{E}^\bullet), v(\mathscr{E}^\bullet)) \geq 2 \geq 0$, therefore the following inequality holds:

$$|(\beta, v(\mathscr{E}^\bullet)) - (\alpha, v(\mathscr{E}^\bullet))| < \varepsilon |(\alpha, v(\mathscr{E}^\bullet))|$$

Theorem 2.4.10 allows us to conclude that for small enough $\varepsilon$, the map

$$\pi|_{C_\varepsilon(\sigma)} : C_\varepsilon(\sigma) \longrightarrow B_\varepsilon(\alpha)$$

is onto, and hence, by Theorem 4.1.11 and Proposition 2.4.9, a homeomorphism. It follows from this that every stability condition is full. Fix a positive real number $\eta < \frac{1}{8}$, and suppose that $\varepsilon < \frac{\sin(\pi\eta)}{2}$. Then, by Theorem 2.4.10 and Lemma

for each $\sigma \in \pi^{-1}(\alpha)$, the subset $C_\varepsilon(\sigma)$ is mapped homeomorphically by $\pi$ onto $B_\varepsilon(\sigma)$. The only thing we need to check is that there is a disjoint union

$$\pi^{-1}(B_\varepsilon(\alpha)) = \bigcup_{\sigma\in\pi^{-1}(\alpha)} C_\varepsilon(\sigma).$$

This simply follows. Take $\tau \in \pi^{-1}(B_\varepsilon(\alpha))$, i.e. a stability condition $\tau$ such that $\pi(\tau) = \beta \in B_\varepsilon(\alpha)$. If $\mathscr{E}^\bullet$ is $\tau$-stable, then

$$|(\beta, v(\mathscr{E}^\bullet)) - (\alpha, v(\mathscr{E}^\bullet))| < \varepsilon|(\alpha, v(\mathscr{E}^\bullet))|$$

which implies that

$$|(\beta, v(\mathscr{E}^\bullet)) - (\alpha, v(\mathscr{E}^\bullet))| < \frac{\varepsilon}{1-\varepsilon}|(\beta, v(\mathscr{E}^\bullet))| < 2\varepsilon|(\beta, v(\mathscr{E}^\bullet))|.$$

Applying Theorem 2.4.10 again gives that there is a stability condition $\sigma \in \pi^{-1}(\alpha)$ such that $d(\sigma,\tau) < \eta$, and then $\tau \in C_\varepsilon(\sigma)$.

□

**Definition 4.3.4.** A connected component $\mathrm{Stab}^*(X) \subset \mathrm{Stab}(X)$ is called *good* if it contains a stability condition $\sigma$ such that $\pi(\sigma) \in \mathscr{P}_0(X)$. A stability condition is called good if it lies in a good connected component

*Remark* 4.3.5. A good connected component is also full. Indeed, by Proposition 4.3.3, the image of the map

$$\pi : \mathrm{Stab}^*(X) \longrightarrow \mathscr{N}(X) \otimes \mathbb{C}$$

contains one of the two connected components of $\mathscr{P}_0(X)$, which is not contained in any linear subspace of $\mathscr{N}(X) \otimes \mathbb{C}$.

*Remark* 4.3.6. $\mathrm{Aut}\mathscr{D}(X)$ acts on the set of good stability conditions, because it acts as an isometry preserving thus the fact that the image of a stability condition lies in $\mathscr{P}_0(X)$.

## 4.4 Wall-and-chamber

What we want to show in this section is that if a connected component $\mathrm{Stab}^*(X) \subset \mathrm{Stab}(X)$ is good, then it has a wall-and chamber structure. More precisely, if we consider a compact subset $B \subset \mathrm{Stab}^*(X)$ and a finite set of objects $S \subset \mathscr{D}(X)$, then there is a finite collection of walls, i.e., submanifolds of codimension one, such that if one deforms a stability condition $\sigma \in \mathrm{Stab}^*(X)$, then one of the elements if $S$ which is stable can only become unstable as $\sigma$ crosses one of the walls. However, the set $S$ needs not to be finite: the hypotesis can be weakened by simply asking that its elements have *bounded mass.*

**Definition 4.4.1.** A set of objects $S \subset \mathscr{D}(X)$ is said to have a *bounded mass* in a connected component $\mathrm{Stab}^*(X) \subset \mathrm{Stab}(X)$ if

$$\sup\{m_\sigma(\mathscr{E}^\bullet) \mid \mathscr{E}^\bullet \in S\} < +\infty$$

for some $\sigma = (Z, \mathscr{P}) \in \mathrm{Stab}^*(X)$. The mass of $\mathscr{E}^\bullet$ is defined to be the positive real number $m_\sigma(\mathscr{E}^\bullet) = \sum_i |Z(\mathscr{A}_i^\bullet)|$, where the $\mathscr{A}_i^\bullet$'s are the semistable quotient in the Harder-Narasimhan filtration of $\mathscr{E}^\bullet$. Note that the fact that $\mathrm{Stab}^*(X)$ is connected implies that $d(\sigma, \tau) < +\infty$ for all $\sigma, \tau \in \mathrm{Stab}^*(X)$, so if that condition holds for some $\sigma \in \mathrm{Stab}^*(X)$, then it holds for all the stability condition in that connected component.

The following lemma is very important:

**Lemma 4.4.2.** *Suppose that the subset $S \in \mathscr{D}(X)$ has bounded mass in some good component* $\mathrm{Stab}^*(X)$. *Then the set*

$$\{v(\mathscr{E}^\bullet) \mid \mathscr{E}^\bullet \in S\}$$

*is finite.*

*Proof.* Simply use the definition of mass: by assumption, there exist a $\sigma \in \mathrm{Stab}^*(X)$ such that $\pi(\sigma) \in \mathscr{P}_0(X)$ and an $m > 0$ such that $m_\sigma(\mathscr{E}^\bullet) < m$ for each $\mathscr{E}^\bullet \in S$. This implies that

$$m_\sigma(\mathscr{E}^\bullet) = \sum_i |Z(\mathscr{A}_i^\bullet)| < m.$$

By the fact that for each stable object $\mathscr{A}^\bullet$ one has $v(\mathscr{A}^\bullet)^2 \geq 2$ (Proposition 4.1.6) and Lemma ???
it follows that there are only finitely many possibilities for the Mukai vectors of the semistable factors, then there are also only finitely many possibilities for the Mukai vector of $\mathscr{E}^\bullet$. □

Now we can better analyze the wall-and-chamber structure which we have announced in the introduction:

**Proposition 4.4.3.** *Let $S \in \mathscr{D}(X)$ be a set of objects with bounded mass in a good connected component* $\mathrm{Stab}^*(X) \subset \mathrm{Stab}(X)$ *and $B \subset$* $\mathrm{Stab}^*(X)$ *be a compact. Then there is a finite collection:*

$$\{W_\gamma\}_{\gamma \in \Gamma}$$

*of real codimension-one submanifolds of* $\mathrm{Stab}^*(X)$, *which need not to be closed, such that any connected component*

$$C \subset B \setminus \bigcup_{\gamma \in \Gamma} W_\gamma$$

*has the following property: if $\mathscr{E}^\bullet \in S$ is semistable in $\sigma$ for some $\sigma \in C$, then $\mathscr{E}^\bullet$ is also stable for all $\tau \in C$ and if, moreover, $\mathscr{E}^\bullet \in S$ has primitive Mukai vector (i.e., $v(\mathscr{E}^\bullet)\mathbb{Z}$ is a primitive sublattice of the lattice associated to $H^*(X, \mathbb{Z})$), then $\mathscr{E}^\bullet$ is $\tau$-stable for all $\tau \in C$.*

*Proof.* Define

$$T \subset \mathscr{D}(X) := \{0 \neq \mathscr{A}^\bullet \in \mathscr{D}(X) \mid m_\sigma(\mathscr{A}^\bullet) \leq m_\sigma(\mathscr{E}^\bullet) \text{ for some } \sigma \in B \text{ and } \mathscr{E}^\bullet \in S\}.$$

By the fact that $B$ is compact, we can say that the quotient $\frac{m_\tau(\mathscr{E}^\bullet)}{m_\sigma(\mathscr{E}^\bullet)}$ is uniformly bounded for each $\sigma, \tau \in B$, so the subset $T \subset \mathscr{D}(X)$ has bounded mass in $\mathrm{Stab}^*(X)$. Notice that all the semistable factors of the elements in $S$ lie in $T$. Now, applying the previous Lemma, we can consider the finite set

$$\{v_i,\ i \in I \mid v_i = v(\mathscr{E}^\bullet) \text{ for some } \mathscr{E}^\bullet \in T\}$$

and the set of indices

$$\Gamma = \{(i,j) \in I \times I \mid v_i \text{ and } v_j \text{ do not lie on the same real line in } \mathscr{N}(X) \otimes \mathbb{R}\}.$$

Finally, take $\gamma = (i,j) \in \Gamma$ and define

$$W_\gamma = \left\{ \sigma = (Z, \mathscr{P}) \in \mathrm{Stab}^*(X) \mid \frac{Z(v_i)}{Z(v_j)} \in \mathbb{R}_{>0} \right\}.$$

Since $\mathrm{Stab}^*(X)$ is a good component, which implies that it is also full, then the map $\pi : \mathrm{Stab}^*(X) \longrightarrow \mathscr{N}(X) \otimes \mathbb{C}$ is a local homeomorphism. Since $W_\gamma$ is the inverse image under $\pi$ of an open subset of a real quadric in $\mathscr{N}(X) \otimes \mathbb{C}$, then it is a real codimension-one submanifold of $\mathrm{Stab}^*(X)$.
Now, if $C \subset B$ is a connected component of

$$B \setminus \bigcup_{\gamma \in \Gamma} W_\gamma,$$

and $\mathscr{E}^\bullet \in S$, we can consider a subset $V \subset C$

$$V := \{\sigma \in C \mid \mathscr{E}^\bullet \text{ is } \sigma - \text{semistable } \}$$

assuming $V$ to be nonempty. We want to show that $V$ is open in $C$, and that $V$ coincides with $C$. Suppose that $\sigma = (Z, \mathscr{P}) \in V$ and $\mathscr{E}^\bullet \in \mathscr{P}(\phi)$ for some $\phi \in \mathbb{R}$. Now, take $\eta \in (0, \frac{1}{8})$ such that the open neighborhood

$$U = \{\tau \in \mathrm{Stab}(X) \mid d(\sigma, \tau) < \eta\}$$

is entirely contained in $C$. The assumption that $\eta \in (0, \frac{1}{8})$ tells us that if $\mathscr{A}^\bullet$ is a semistable factor of $\mathscr{E}^\bullet$ for some stability condition belonging to $U$, then $\mathscr{A}^\bullet$ lies in the abelian category

$$\mathscr{A} = \mathscr{P}\left(\left(\phi - \frac{1}{2}, \phi + \frac{1}{2}\right]\right) \subset \mathscr{D}(X)$$

and that the central charge of $\mathscr{E}^\bullet$ with respect to some $\tau = (W, \mathscr{Q}) \in U$ lies in the half-plane

$$H_\phi = \left\{ re^{i\pi\psi} \mid r > 0 \text{ and } \phi - \frac{1}{2} < \psi < \phi + \frac{1}{2} \right\}.$$

Suppose that $\mathscr{E}^\bullet$ is unstable for some $\sigma' = (Z', \mathscr{P}') \in U$. Then there is a semistable factor $\mathscr{A}^\bullet$ of $\mathscr{E}^\bullet$ with respect to $\sigma'$ which is a subobject of $\mathscr{E}^\bullet$ in the category $\mathscr{A}$ and which satisfies $\Im \frac{Z'(\mathscr{A}^\bullet)}{Z'(\mathscr{E}^\bullet)} > 0$. As $\tau$ varies in $U$, then the complex numbers $W(\mathscr{A}^\bullet)$ and $W(\mathscr{E}^\bullet)$ remain in $H_\phi$ and therefore, since $U$ is a connected component of

$$B \setminus \bigcup_{\gamma \in \Gamma} W_\gamma,$$

and $\mathscr{A}^\bullet$, $\mathscr{E}^\bullet \in T$, then $\Im \frac{W(\mathscr{A}^\bullet)}{W(\mathscr{E}^\bullet)} > 0$ for all $\tau = (W, \mathscr{Q})$, which contradicts the fact that $\mathscr{E}^\bullet$ is $\sigma$-semistable. Now, suppose that $\mathscr{E}^\bullet \in S$ has primitive Mukai vector, and suppose that $\mathscr{E}^\bullet$ is semistable but not stable for some $\sigma \in B$. Each stable factor of $\mathscr{E}^\bullet$ has mass less then $\mathscr{E}^\bullet$, so its Mukai vector belongs to the set

$$\{v_i,\ i \in I \mid v_i = v(\mathscr{E}^\bullet) \text{ for some } \mathscr{E}^\bullet \in T\}.$$

Since $v(\mathscr{E}^\bullet)$ is primitive, not all the Mukai vectors of the stable factors of $\mathscr{E}^\bullet$ are multiples of the Mukai vector of $\mathscr{E}^\bullet$ But the phases of all the stable factors are the same, so one must have $\sigma \in W_\gamma$ for some $\gamma = (i, j) \in \Gamma$. $\square$

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